A Complete Solution to Problems in "An Introduction to Quantum Field Theory" by Peskin and Schroeder

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Preface

In this note I provide solutions to all problems and final projects in the book *An Introduction to Quantum Field Theory* by M. E. Peskin and D. V. Schroeder [1], which I worked out and typed into TEX during the first two years of my PhD study at Tsinghua University. I once posted a draft version of them on my personal webpage using a server provided by Tsinghua, which was however closed unfortunately after I graduated. Since then I received quite a number of emails asking for the solutions, so I decided to put them on arXiv.*

Nothing much has been updated in this note compared with the previous draft due to the lack of time, except for some editorial work, as well as a few newly added references. In particular, I don't have enough time to proofread and therefore I cannot guarantee the correctness of them, though I expect that most of them are correct. With that said, any feedback via email† about errors, either physical or typographical, is much appreciated.

I would not claim any novelty or originality of this note, since almost all of problems in the book belong to standard material of quantum field theory. Occasionally, I learned the answer to a problem or the strategy for solving it before I started to work it out. But still, I believe that the problem set in the book will always remain a treasure to any beginner of this subject, and I feel it worthy to write up the solutions.

The contraction macro provided by the authors of the book‡ has been used in this note.

I would like to express my gratitude to Prof. Qing Wang and Prof. Hong-Jian He for their wonderful courses of quantum field theory and their great help in my early days of learning this subject. I would also like to thank Prof. Michael Peskin in particular, for his generous permission and kind encouragement to letting me publish this note.

Comments on notations. All notations and conventions are the same with the book. The book will be cited in the main text as "P&S" for short. The $+i\epsilon$ prescription for Feynman propagators is always assumed and is usually hidden.

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*The submission, however, was rejected by one of arXiv volunteer moderators based on the reason that "arXiv does not allow submissions containing solutions to problems in physics textbooks", and that "(the) moderators consider that this type of submissions are harmful for students and instructors". Insofar as I can see, however, the solution can only do harm to those who are willing to do harm to themselves.

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‡http://physics.weber.edu/schroeder/qftbook.html

Contents

Chapter 2

The Klein-Gordon Field

2.1 Classical electromagnetism

In this problem we derive the field equations and energy-momentum tensor from the following action of classical electrodynamics,

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}, \qquad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.1}$$

(a) Maxwell's equations To take variation of the classical action with respect to the field A_μ, we note,

$$\frac{\delta F_{\mu\nu}}{\delta(\partial_\lambda A_\kappa)} = \delta^\lambda_\mu \delta^\kappa_\nu - \delta^\lambda_\nu \delta^\kappa_\mu, \qquad \frac{\delta F_{\mu\nu}}{\delta A_\lambda} = 0. \tag{2.2}$$

Then from the first equality we get:

$$\frac{\delta}{\delta(\partial_\lambda A_\kappa)}\big(F_{\mu\nu}F^{\mu\nu}\big) = 4F^{\lambda\kappa}. \tag{2.3}$$

Now substitute this into Euler-Lagrange equation, we have,

$$0 = \partial_\mu\Big(\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_\nu)}\Big) - \frac{\delta\mathcal{L}}{\delta A_\nu} = -\partial_\mu F^{\mu\nu}. \tag{2.4}$$

This is sometimes called the "second pair" of Maxwell's equations. The so-called "first pair" follows directly from the definition of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and reads

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\mu\lambda} = 0. \tag{2.5}$$

The familiar electric and magnetic field strengths can be written as $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$, respectively. From this we deduce the Maxwell's equations in terms of E^i and B^i:

$$\partial^i E^i = 0, \qquad \epsilon^{ijk}\partial^j B^k - \partial^0 E^i = 0, \qquad \epsilon^{ijk}\partial^j E^k = 0, \qquad \partial^i B^i = 0. \tag{2.6}$$

(b) The energy-momentum tensor The energy-momentum tensor can be defined to be the Nöther current of the space-time translational symmetry. Under space-time translation the vector A_μ transforms as,

$$\delta^\mu A^\nu = \partial^\mu A^\nu. \tag{2.7}$$

Thus

$$\tilde{T}^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\lambda)}\partial^\nu A_\lambda - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa}. \tag{2.8}$$

Obviously, this tensor is not symmetric. We can add an additional term $\partial_\lambda K^{\lambda\mu\nu}$ to $\tilde{T}^{\mu\nu}$ with $K^{\lambda\mu\nu}$ antisymmetric with its first two indices. It's easy to see that this term does not affect the conservation of $\tilde{T}^{\mu\nu}$. So if we choose $K^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, then,

$$T^{\mu\nu} = \tilde{T}^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\kappa}F^{\lambda\kappa}. \tag{2.9}$$

Now this tensor is symmetric and is sometimes called the Belinfante tensor in literature. We can also rewrite it in terms of E^i and B^i,

$$T^{00} = \frac{1}{2}(E^iE^i + B^iB^i), \qquad T^{i0} = T^{0i} = \epsilon^{ijk}E^jB^k, \quad \text{etc.} \tag{2.10}$$

2.2 The complex scalar field

The Lagrangian is given by,

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{2.11}$$

(a) The conjugate momenta of ϕ and ϕ^*:

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^*, \qquad \tilde{\pi} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} = \dot{\phi} = \pi^*. \tag{2.12}$$

The canonical commutation relations:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = [\phi^*(\mathbf{x}), \pi^*(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}), \tag{2.13}$$

The rest of commutators are all zero.

The Hamiltonian:

$$H = \int \mathrm{d}^3x \left(\pi\dot{\phi} + \pi^*\dot{\phi}^* - \mathcal{L}\right) = \int \mathrm{d}^3x \left(\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\right). \tag{2.14}$$

(b) Now we Fourier transform the field ϕ as:

$$\phi(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\Big(a_\mathbf{p}e^{-ip\cdot x} + b_\mathbf{p}^\dagger e^{ip\cdot x}\Big), \tag{2.15}$$

thus:

$$\phi^*(x) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\Big(b_\mathbf{p}e^{-ip\cdot x} + a_\mathbf{p}^\dagger e^{ip\cdot x}\Big). \tag{2.16}$$

Substitute the mode expansion into the Hamiltonian:

$$
\begin{aligned}
H &= \int \mathrm{d}^3x\,\big(\dot{\phi}^*\dot{\phi} + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi\big)\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\\
&\quad\times\bigg[E_{\mathbf{p}}E_{\mathbf{q}}\Big(a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x} - b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} - b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\\
&\quad + \mathbf{p}\cdot\mathbf{q}\Big(a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x} - b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} - b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\\
&\quad + m^2\Big(a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x} + b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x} + b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\bigg]\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\\
&\quad\times\bigg[(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2)\Big(a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{\mathrm{i}(p-q)\cdot x} + b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-\mathrm{i}(p-q)\cdot x}\Big)\\
&\quad - (E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} - m^2)\Big(b_{\mathbf{q}}a_{\mathbf{q}}e^{-\mathrm{i}(p+q)\cdot x} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{\mathrm{i}(p+q)\cdot x}\Big)\bigg]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\\
&\quad\times\bigg[(E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} + m^2)\Big(a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t} + b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&\quad - (E_{\mathbf{p}}E_{\mathbf{q}} + \mathbf{p}\cdot\mathbf{q} - m^2)\Big(b_{\mathbf{q}}a_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t} + a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\bigg]\\
&= \int \mathrm{d}^3x\,\frac{E_{\mathbf{p}}^2 + \mathbf{p}^2 + m^2}{2E_{\mathbf{p}}}\big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}b_{\mathbf{p}}^\dagger\big)\\
&= \int \mathrm{d}^3x\,E_{\mathbf{p}}\big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + [b_{\mathbf{p}}, b_{\mathbf{p}}^\dagger]\big),
\end{aligned}
\tag{2.17}
$$

where we have used the mass-shell condition $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. Note that the last term contributes an infinite constant, which can be interpreted as the vacuum energy and can be dropped, for instance, by the prescription of normal ordering. Then we get a finite Hamiltonian,

$$
H = \int \mathrm{d}^3x\,E_{\mathbf{p}}\big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\big), \tag{2.18}
$$

Hence we get two sets of particles with the same mass m.

(c) The theory is invariant under the global transformation: $\phi \to e^{i\theta}\phi$, $\phi^* \to e^{-i\theta}\phi^*$. The corresponding conserved charge is:

$$
Q = i\int \mathrm{d}^3x\,\big(\phi^*\dot{\phi} - \dot{\phi}^*\phi\big). \tag{2.19}
$$

Rewrite this in terms of the creation and annihilation operators:

$$
Q = \mathrm{i}\int \mathrm{d}^3x\,\big(\phi^*\dot{\phi} - \dot{\phi}^*\phi\big)
$$

$$\begin{aligned}
&= \mathrm{i}\int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\bigg[\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x}\Big)\frac{\partial}{\partial t}\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\\
&\quad-\frac{\partial}{\partial t}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x}\Big)\cdot\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\bigg]\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\bigg[E_{\mathbf{q}}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}+a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}-b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\\
&\quad-E_{\mathbf{p}}\Big(b_{\mathbf{p}}e^{-\mathrm{i}p\cdot x}-a_{\mathbf{p}}^\dagger e^{\mathrm{i}p\cdot x}\Big)\Big(a_{\mathbf{q}}e^{-\mathrm{i}q\cdot x}+b_{\mathbf{q}}^\dagger e^{\mathrm{i}q\cdot x}\Big)\bigg]\\
&= \int \mathrm{d}^3x \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\bigg[(E_{\mathbf{q}}-E_{\mathbf{p}})\Big(b_{\mathbf{p}}a_{\mathbf{q}}e^{-\mathrm{i}(p+q)\cdot x}-a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{\mathrm{i}(p+q)\cdot x}\Big)\\
&\quad+(E_{\mathbf{q}}+E_{\mathbf{p}})\Big(a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{\mathrm{i}(p-q)\cdot x}-b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-\mathrm{i}(p-q)\cdot x}\Big)\bigg]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}}\frac{\mathrm{d}^3q}{(2\pi)^3\sqrt{2E_{\mathbf{q}}}}\\
&\quad\times\bigg[(E_{\mathbf{q}}-E_{\mathbf{p}})\Big(b_{\mathbf{p}}a_{\mathbf{q}}e^{-\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}})t}-a_{\mathbf{p}}^\dagger b_{\mathbf{q}}^\dagger e^{\mathrm{i}(E_{\mathbf{p}}+E_{\mathbf{q}}t)}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})\\
&\quad+(E_{\mathbf{q}}+E_{\mathbf{p}})\Big(a_{\mathbf{p}}^\dagger a_{\mathbf{q}}e^{\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}-b_{\mathbf{p}}b_{\mathbf{q}}^\dagger e^{-\mathrm{i}(E_{\mathbf{p}}-E_{\mathbf{q}})t}\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\bigg]\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3 2E_{\mathbf{p}}}\cdot 2E_{\mathbf{p}}(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}-b_{\mathbf{p}}b_{\mathbf{p}}^\dagger)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\big(a_{\mathbf{p}}^\dagger a_{\mathbf{p}}-b_{\mathbf{p}}^\dagger b_{\mathbf{p}}\big),
\end{aligned} \tag{2.20}$$

where the last equal sign holds up to an infinitely large constant term, as we did when calculating the Hamiltonian in (b). Then the commutators follow straightforwardly:

$$[Q,a^\dagger]=a^\dagger,\qquad [Q,b^\dagger]=-b^\dagger. \tag{2.21}$$

We see that the particle a carries one unit of positive charge, and b carries one unit of negative charge.

(d) Now we consider the case with two complex scalars of same mass. In this case the Lagrangian is given by

$$\mathcal{L}=\partial_\mu\Phi_i^\dagger\partial^\mu\Phi_i-m^2\Phi_i^\dagger\Phi_i, \tag{2.22}$$

where Φ_i with $i=1,2$ is a two-component complex scalar. Then it is straightforward to see that the Lagrangian is invariant under the $U(2)$ transformation $\Phi_i\to U_{ij}\Phi_j$ with U_{ij} a matrix in fundamental representation of $U(2)$ group. The $U(2)$ group, locally isomorphic to $SU(2)\times U(1)$, is generated by 4 independent generators 1 and $\frac{1}{2}\tau^a$, with τ^a Pauli matrices. Then 4 independent Nöther currents are associated, which are given by,

$$j_\mu=-\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i)}\Delta\Phi_i-\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i^*)}\Delta\Phi_i^*=-(\partial_\mu\Phi_i^*)(\mathrm{i}\Phi_i)-(\partial_\mu\Phi_i)(-\mathrm{i}\Phi_i^*),$$

$$j^a_\mu = -\frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi_i)}\Delta^a\Phi_i - \frac{\partial\mathcal{L}}{\partial(\partial^\mu\Phi^*_i)}\Delta^a\Phi^*_i = -\frac{\mathrm{i}}{2}\Big[(\partial_\mu\Phi^*_i)\tau_{ij}\Phi_j - (\partial_\mu\Phi_i)\tau_{ij}\Phi^*_j\Big]. \tag{2.23}$$

The overall sign is chosen such that the particle carry positive charge, as will be seen in the following. Then the corresponding Nöther charges are given by,

$$\begin{aligned} Q =& -\mathrm{i}\int \mathrm{d}^3x\,\big(\dot\Phi^*_i\Phi_i - \Phi^*_i\dot\Phi_i\big), \\ Q^a =& -\frac{\mathrm{i}}{2}\int \mathrm{d}^3x\,\big[\dot\Phi^*_i(\tau^a)_{ij}\Phi_j - \Phi^*_i(\tau^a)_{ij}\dot\Phi_j\big]. \end{aligned} \tag{2.24}$$

Repeating the derivations above, we can also rewrite these charges in terms of creation and annihilation operators, as,

$$\begin{aligned} Q =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{p}} - b^\dagger_{i\mathbf{p}}b_{i\mathbf{p}}\Big), \\ Q^a =& \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}\tau^a_{ij}a_{i\mathbf{p}} - b^\dagger_{j\mathbf{p}}\tau^a_{ij}b_{j\mathbf{p}}\Big). \end{aligned} \tag{2.25}$$

The generalization to N-component complex scalar is straightforward. In this case we only need to replace the generators $\tau^a/2$ of $SU(2)$ group to the generators t^a in the fundamental representation of $SU(N)$ group with commutation relation $[t^a,t^b] = \mathrm{i}f^{abc}t^c$.

Then we are ready to calculate the commutators among all these Nöther charges and the Hamiltonian. Firstly we show that all charges of the $U(N)$ group commute with the Hamiltonian. For the $U(1)$ generator, we have

$$\begin{aligned} [Q,H] =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_\mathbf{q}\Big[\big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{p}} - b^\dagger_{i\mathbf{p}}b_{i\mathbf{p}}\big),\big(a^\dagger_{j\mathbf{q}}a_{j\mathbf{q}} + b^\dagger_{j\mathbf{q}}b_{j\mathbf{q}}\big)\Big] \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_\mathbf{q}\Big(a^\dagger_{i\mathbf{p}}[a_{i\mathbf{p}},a^\dagger_{j\mathbf{q}}]a_{j\mathbf{q}} + a^\dagger_{j\mathbf{q}}[a^\dagger_{i\mathbf{p}},a_{j\mathbf{q}}]a_{i\mathbf{p}} \;+\; (a\to b)\Big) \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,E_\mathbf{q}\Big(a^\dagger_{i\mathbf{p}}a_{i\mathbf{q}} - a^\dagger_{i\mathbf{q}}a_{i\mathbf{p}} \;+\; (a\to b)\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ =& \,0. \end{aligned} \tag{2.26}$$

Similar calculation gives $[Q^a,H]=0$. Then we consider the commutation among internal $U(N)$ charges:

$$\begin{aligned} [Q^a,Q^b] =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,\Big[\big(a^\dagger_{i\mathbf{p}}t^a_{ij}a_{j\mathbf{p}} - b^\dagger_{i\mathbf{p}}t^a_{ij}b_{j\mathbf{p}}\big),\big(a^\dagger_{k\mathbf{q}}t^b_{k\ell}a_{\ell\mathbf{q}} - b^\dagger_{k\mathbf{q}}t^b_{k\ell}b_{\ell\mathbf{q}}\big)\Big] \\ =& \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\,\Big(a^\dagger_{i\mathbf{p}}t^a_{ij}t^b_{j\ell}a_{\ell\mathbf{q}} - a^\dagger_{k\mathbf{q}}t^b_{k\ell}t^a_{\ell j}a_{j\mathbf{p}} \;+\; (a\to b)\Big)(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ =& \,\mathrm{i}f^{abc}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\Big(a^\dagger_{i\mathbf{p}}t^c_{ij}a_{j\mathbf{p}} - b^\dagger_{i\mathbf{p}}t^c_{ij}b_{j\mathbf{p}}\Big) \\ =& \,\mathrm{i}f^{abc}Q^c, \end{aligned} \tag{2.27}$$

and similarly, $[Q,Q] = [Q^a,Q] = 0$.

2.3 The spacelike correlation function

We evaluate the correlation function of a scalar field at two points,

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle, \tag{2.28}$$

with $x-y$ being spacelike. Since any spacelike interval $x-y$ can be transformed to a form such that $x^0 - y^0 = 0$, thus we will simply take:

$$x^0 - y^0 = 0, \qquad \text{and} \qquad |\mathbf{x}-\mathbf{y}|^2 = r^2 > 0. \tag{2.29}$$

Now:

$$\begin{aligned} D(x-y) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2\sqrt{m^2+p^2}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\ &= \frac{1}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\varphi \int_{-1}^{1} \mathrm{d}\cos\theta \int_0^\infty \mathrm{d}p \frac{p^2}{2\sqrt{m^2+p^2}} e^{ipr\cos\theta} \\ &= \frac{-\mathrm{i}}{2(2\pi)^2 r} \int_{-\infty}^{\infty} \mathrm{d}p \frac{pe^{\mathrm{i}pr}}{\sqrt{m^2+p^2}} \end{aligned} \tag{2.30}$$

Now we make the path deformation on p-complex plane, as is shown in Figure 2.3 of P&S. Then the integral becomes,

$$D(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty \mathrm{d}\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}} = \frac{m}{4\pi^2 r} \mathrm{K}_1(mr). \tag{2.31}$$

Chapter 3

The Dirac Field

3.1 Lorentz group

The generators of Lorentz group satisfy the following commutation relation,

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathrm{i}(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}). \tag{3.1}$$

(a) Let us redefine the generators as $L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}$ (All Latin indices denote spatial components), with L^i generate rotations, and K^i generate boosts. The commutators of them can be derived straightforwardly to be,

$$[L^i, L^j] = \mathrm{i}\epsilon^{ijk}L^k, \qquad [K^i, K^j] = -\mathrm{i}\epsilon^{ijk}L^k. \tag{3.2}$$

If we further define $J^i_\pm = \frac{1}{2}(L^i \pm \mathrm{i}K^i)$, then the commutators become,

$$[J^i_\pm, J^j_\pm] = \mathrm{i}\epsilon^{ijk}J^k_\pm, \qquad [J^i_+, J^j_-] = 0. \tag{3.3}$$

Thus we see that the algebra of the Lorentz group is a direct sum of two identical algebra $\mathfrak{su}(2)$.

(b) It follows that we can classify the finite dimensional representations of the Lorentz group by a pair (j_+, j_-), where $j_\pm = 0, 1/2, 1, 3/2, 2, \cdots$ are labels of irreducible representations of $SU(2)$.

We study two specific cases.

1. $(\frac{1}{2}, 0)$. Following the definition, we have J^i_+ represented by $\frac{1}{2}\sigma^i$ and J^i_- represented by 0. This implies

$$L^i = (J^i_+ + J^i_-) = \tfrac{1}{2}\sigma^i, \qquad K^i = -\mathrm{i}(J^i_+ - J^i_-) = -\tfrac{\mathrm{i}}{2}\sigma^i. \tag{3.4}$$

Hence a field ψ under this representation transforms as:

$$\psi \to e^{-\mathrm{i}\theta^i\sigma^i/2 - \eta^i\sigma^i/2}\psi. \tag{3.5}$$

2. $(\frac{1}{2},0)$. In this case, $J_+^i \to 0$, $J_-^i \to \frac{1}{2}\sigma^i$. Then

$$L^i = (J_+^i + J_-^i) = \tfrac{1}{2}\sigma^i, \qquad K^i = -\mathrm{i}(J_+^i - J_-^i) = \tfrac{\mathrm{i}}{2}\sigma^i. \tag{3.6}$$

Hence a field ψ under this representation transforms as:

$$\psi \to e^{-\mathrm{i}\theta^i\sigma^i/2+\eta^i\sigma^i/2}\psi. \tag{3.7}$$

We see that a field under the representation $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are precisely the left-handed spinor ψ_L and right-handed spinor ψ_R, respectively.

(c) Let us consider the case of $(\frac{1}{2},\frac{1}{2})$. To put the field associated with this representation into a familiar form, we note that a left-handed spinor can also be rewritten as row, which transforms under the Lorentz transformation as:

$$\psi_L^T\sigma^2 \ \to\ \psi_L^T\sigma^2\big(1+\tfrac{\mathrm{i}}{2}\theta^i\sigma^i+\tfrac{1}{2}\eta^i\sigma^i\big). \tag{3.8}$$

Then the field under the representation $(\frac{1}{2},\frac{1}{2})$ can be written as a tensor with spinor indices:

$$\psi_R\psi_L^T\sigma^2 \equiv V^\mu\bar{\sigma}_\mu = \begin{pmatrix} V^0+V^3 & V^1-\mathrm{i}V^2 \\ V^1+\mathrm{i}V^2 & V^0-V^3 \end{pmatrix}. \tag{3.9}$$

In what follows we will prove that V^μ is in fact a Lorentz vector.

A quantity V^μ is called a Lorentz vector, if it satisfies the following transformation law:

$$V^\mu \to \Lambda^\mu{}_\nu V^\nu, \tag{3.10}$$

where $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}(J^{\rho\sigma})^\mu{}_\nu$ in its infinitesimal form. We further note that:

$$(J^{\rho\sigma})_{\mu\nu} = \mathrm{i}(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu). \tag{3.11}$$

and also, $\omega_{ij} = \epsilon_{ijk}\theta^k$, $\omega_{0i} = -\omega_{i0} = \eta^i$, then the combination $V^\mu\bar{\sigma}_\mu = V^i\sigma^i + V^0$ transforms according to

$$\begin{aligned}
V^i\sigma^i \to& \Big(\delta^i_j - \frac{\mathrm{i}}{2}\omega_{mn}(J^{mn})^i{}_j\Big)V^j\sigma^i + \Big(-\frac{\mathrm{i}}{2}\omega_{0n}(J^{0n})^i{}_0 - \frac{\mathrm{i}}{2}\omega_{n0}(J^{n0})^0{}_i\Big)V^0\sigma^i \\
=& \Big(\delta^i_j - \tfrac{\mathrm{i}}{2}\epsilon_{mnk}\theta^k(-\mathrm{i})(\delta^m_i\delta^n_j - \delta^m_j\delta^n_i)\Big)V^j\sigma^i + \big(-\mathrm{i}\eta^i(-\mathrm{i})(-\delta^n_i)\big)V^0\sigma^i \\
=& V^i\sigma^i - \epsilon^{ijk}V^i\theta^j\sigma^k + V^0\eta^i\sigma^i,
\end{aligned}$$

$$\begin{aligned}
V^0 \to& V^0 + \Big(-\frac{\mathrm{i}}{2}\omega_{0n}(J^{0n})_{0i} - \frac{\mathrm{i}}{2}\omega_{n0}(J^{n0})_{0i}\Big)V^i \\
=& V^0 + \big(-\mathrm{i}\eta^i(i\delta^n_i)\big)V^i = V^0 + \eta^i V^i.
\end{aligned}$$

In total, we have

$$V^\mu\bar{\sigma}_\mu \to \big(\sigma^i - \epsilon^{ijk}\theta^j\sigma^k + \eta^i\big)V^i + (1+\eta^i\sigma^i)V^0. \tag{3.12}$$

If we can reach the same conclusion by treating the combination $V^\mu\bar{\sigma}_\mu$ a matrix transforming under the representation $(\frac{1}{2},\frac{1}{2})$, then our original statement will be proved. In fact:

$$\begin{aligned} V^\mu\bar{\sigma}_\mu \to & \Big(1-\frac{\mathrm{i}}{2}\theta^j\sigma^j+\frac{1}{2}\eta^j\sigma^j\Big)V^\mu\sigma_\mu\Big(1+\frac{\mathrm{i}}{2}\theta^j\sigma^j+\frac{1}{2}\eta^j\sigma^j\Big) \\ = & \Big(\sigma^i+\frac{\mathrm{i}}{2}\theta^j[\sigma^i,\sigma^j]+\frac{1}{2}\eta^j\{\sigma^i,\sigma^j\}\Big)V^i+(1+\eta^i\sigma^i)V^0 \\ = & \big(\sigma^i-\epsilon^{ijk}\theta^j\sigma^k+\eta^i\big)V^i+(1+\eta^i\sigma^i)V^0, \end{aligned} \tag{3.13}$$

as expected. Hence we proved that V^μ is a Lorentz vector.

3.2 The Gordon identity

In this problem we derive the Gordon identity,

$$\bar{u}(p')\gamma^\mu u(p)=\bar{u}(p')\Big(\frac{p'^\mu+p^\mu}{2m}+\frac{\mathrm{i}\sigma^{\mu\nu}(p'_\nu-p_\nu)}{2m}\Big)u(p). \tag{3.14}$$

Let us start from the right hand side,

$$\begin{aligned} & \bar{u}(p')\Big(\frac{p'^\mu+p^\mu}{2m}+\frac{\mathrm{i}\sigma^{\mu\nu}(p'_\nu-p_\nu)}{2m}\Big)u(p) \\ = & \frac{1}{2m}\bar{u}(p')\Big((p'^\mu+p^\mu)+\mathrm{i}\sigma^{\mu\nu}(p'_\nu-p_\nu)\Big)u(p) \\ = & \frac{1}{2m}\bar{u}(p')\Big(\eta^{\mu\nu}(p'_\nu+p_\nu)-\frac{1}{2}[\gamma^\mu,\gamma^\nu](p'_\nu-p_\nu)\Big)u(p) \\ = & \frac{1}{2m}\bar{u}(p')\Big(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}(p'_\nu+p_\nu)-\frac{1}{2}[\gamma^\mu,\gamma^\nu](p'_\nu-p_\nu)\Big)u(p) \\ = & \frac{1}{2m}\bar{u}(p')\Big(\not{p}'\gamma^\mu+\gamma^\mu\not{p}\Big)u(p)=\bar{u}(p')\gamma^\mu u(p), \end{aligned}$$

where we have used the commutator and anti-commutators of gamma matrices, as well as the Dirac equation.

3.3 The spinor products

In this problem, together with the Problems 5.3 and 5.6, we will develop a formalism that can be used to calculating scattering amplitudes involving massless fermions or vector particles. This method can profoundly simplify the calculations, especially in the calculations of QCD. Here we will derive the basic fact that the spinor products can be treated as the square root of the inner product of lightlike Lorentz vectors. Then, in Problem 5.3 and 5.6, this relation will be put in use in calculating the amplitudes with external spinors and external photons, respectively.

To begin with, let k_0^μ and k_1^μ be fixed four-vectors satisfying $k_0^2=0$, $k_1^2=-1$ and $k_0\cdot k_1=0$. With these two reference momenta, we define the following spinors:

1. Let u_{L0} be left-handed spinor with momentum k_0;

2. Let $u_{R0} = \not{k}_1 u_{L0}$;
3. For any lightlike momentum p ($p^2 = 0$), define:

$$u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{R0}, \qquad u_R(p) = \frac{1}{\sqrt{2p \cdot k}} \not{p} u_{L0}. \tag{3.15}$$

(a) We show that $\not{k}_0 u_{R0} = 0$ and $\not{p} u_L(p) = \not{p} u_R(p) = 0$ for any lightlike p. That is, u_{R0} is a massless spinor with momentum k_0, and $u_L(p)$, $u_R(p)$ are massless spinors with momentum p. This is quite straightforward,

$$\not{k}_0 u_{R0} = \not{k}_0 \not{k}_1 u_{L0} = (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) k_{0\mu} k_{1\nu} u_{L0} = 2k_0 \cdot k_1 u_{L0} - \not{k}_1 \not{k}_0 u_{L0} = 0, \tag{3.16}$$

and, by definition,

$$\not{p} u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p}\not{p} u_{R0} = \frac{1}{\sqrt{2p \cdot k_0}} p^2 u_{R0} = 0. \tag{3.17}$$

In the same way, we can show that $\not{p} u_R(p) = 0$.

(b) Now we choose $k_{0\mu} = (E, 0, 0, -E)$ and $k_{1\mu} = (0, 1, 0, 0)$. Then in the Weyl representation, we have:

$$\not{k}_0 u_{L0} = 0 \quad \Rightarrow \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2E \\ 2E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} u_{L0} = 0. \tag{3.18}$$

Thus u_{L0} can be chosen to be $(0, \sqrt{2E}, 0, 0)^T$, and:

$$u_{R0} = \not{k}_1 u_{L0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} u_{L0} = \begin{pmatrix} 0 \\ 0 \\ -\sqrt{2E} \\ 0 \end{pmatrix}. \tag{3.19}$$

Let $p_\mu = (p_0, p_1, p_2, p_3)$, then:

$$\begin{aligned} u_L(p) &= \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{R0} \\ &= \frac{1}{\sqrt{2E(p_0 + p_3)}} \begin{pmatrix} 0 & 0 & p_0 + p_3 & p_1 - ip_2 \\ 0 & 0 & p_1 + ip_2 & p_0 - p_3 \\ p_0 - p_3 & -p_1 + ip_2 & 0 & 0 \\ -p_1 - ip_2 & p_0 + p_3 & 0 & 0 \end{pmatrix} u_{R0} \\ &= \frac{1}{\sqrt{p_0 + p_3}} \begin{pmatrix} -(p_0 + p_3) \\ -(p_1 + ip_2) \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.20}$$

In the same way, we get:

$$u_R(p) = \frac{1}{\sqrt{p_0+p_3}}\begin{pmatrix} 0 \\ 0 \\ -p_1+ip_2 \\ p_0+p_3 \end{pmatrix}. \tag{3.21}$$

(c) We construct explicitly the spinor product $s(p,q)$ and $t(p,q)$.

$$s(p,q) = \bar{u}_R(p)u_L(q) = \frac{(p_1+ip_2)(q_0+q_3)-(q_1+iq_2)(p_0+p_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}; \tag{3.22}$$

$$t(p,q) = \bar{u}_L(p)u_R(q) = \frac{(q_1-iq_2)(p_0+p_3)-(p_1-ip_2)(q_0+q_3)}{\sqrt{(p_0+p_3)(q_0+q_3)}}. \tag{3.23}$$

It can be easily seen that $s(p,q) = -s(q,p)$ and $t(p,q) = (s(q,p))^*$.

Now we calculate the quantity $|s(p,q)|^2$:

$$\begin{aligned} |s(p,q)|^2 =& \frac{\big(p_1(q_0+q_3)-q_1(p_0+p_3)\big)^2+\big(p_2(q_0+q_3)-q_2(p_0+p_3)\big)^2}{(p_0+p_3)(q_0+q_3)} \\ =&(p_1^2+p_2^2)\frac{q_0+q_3}{p_0+p_3}+(q_1^2+q_2^2)\frac{p_0+p_3}{q_0+q_3}-2(p_1q_1+p_2q_2) \\ =&2(p_0q_0-p_1q_1-p_2q_2-p_3q_3) = 2p\cdot q. \end{aligned} \tag{3.24}$$

Where we have used the lightlike properties $p^2 = q^2 = 0$. Thus we see that the spinor product can be regarded as the square root of the 4-vector dot product for lightlike vectors.

3.4 Majorana fermions

(a) We at first study a two-component massive spinor χ lying in $(\frac{1}{2},0)$ representation, transforming according to $\chi \to U_L(\Lambda)\chi$. It satisfies the following equation of motion:

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi - \mathrm{i}m\sigma^2\chi^* = 0. \tag{3.25}$$

To show this equation is indeed an admissible equation, we need to justify: 1) It is relativistically covariant; 2) It is consistent with the mass-shell condition (namely the Klein-Gordon equation).

To show the condition 1) is satisfied, we note that γ^μ is invariant under the simultaneous transformations of its Lorentz indices and spinor indices. That is $\Lambda^\mu{}_\nu U(\Lambda)\gamma^\nu U(\Lambda^{-1}) = \gamma^\mu$. This implies

$$\Lambda^\mu{}_\nu U_R(\Lambda)\bar{\sigma}^\nu U_L(\Lambda^{-1}) = \bar{\sigma}^\mu,$$

as can be easily seen in chiral basis. Then, the combination $\bar{\sigma}^\mu\partial_\mu$ transforms as $\bar{\sigma}^\mu\partial_\mu \to U_R(\Lambda)\bar{\sigma}^\mu\partial_\mu U_L(\Lambda^{-1})$. As a result, the first term of the equation of motion transforms as

$$\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi \to \mathrm{i}U_R(\Lambda)\bar{\sigma}^\mu\partial_\mu U_L(\Lambda^{-1})U_L(\Lambda)\chi = U_R(\Lambda)\big[\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi\big]. \tag{3.26}$$

To show the full equation of motion is covariant, we also need to show that the second term $\mathrm{i}\sigma^2\chi^*$ transforms in the same way. To see this, we note that in the infinitesimal form,

$$U_L = 1 - i\theta^i\sigma^i/2 - \eta^i\sigma^i/2, \qquad U_R = 1 - i\theta^i\sigma^i/2 + \eta^i\sigma^i/2.$$

Then, under an infinitesimal Lorentz transformation, χ transforms as:

$$\chi \to (1 - i\theta^i\sigma^i/2 - \eta^i\sigma^i/2)\chi, \qquad \Rightarrow \qquad \chi^* \to (1 + i\theta^i\sigma^i/2 - \eta^i\sigma^i/2)\chi^*$$
$$\Rightarrow \qquad \sigma^2\chi^* \to \sigma^2(1 + i\theta^i(\sigma^*)^i/2 - \eta^i(\sigma^*)^i/2)\chi^* = (1 - i\theta^i\sigma^i/2 + \eta^i\sigma^i/2)\sigma^2\chi^*.$$

That is to say, $\sigma^2\chi^*$ is a right-handed spinor that transforms as $\sigma^2\chi^* \to U_R(\Lambda)\sigma^2\chi^*$. Thus we see the the two terms in the equation of motion transform in the same way under the Lorentz transformation. In other words, this equation is Lorentz covariant.

To show the condition 2) also holds, we take the complex conjugation of the equation:

$$-\mathrm{i}(\bar{\sigma}^*)^\mu\partial_\mu\chi^* - \mathrm{i}m\sigma^2\chi = 0.$$

Combining this and the original equation to eliminate χ^*, we get

$$(\partial^2 + m^2)\chi = 0, \tag{3.27}$$

which has the same form with the Klein-Gordon equation.

(b) Now we show that the equation of motion above for the spinor χ can be derived from the following action through the variation principle:

$$S = \int \mathrm{d}^4x \left[\chi^\dagger \mathrm{i}\bar{\sigma}\cdot\partial\chi + \frac{im}{2}(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*)\right]. \tag{3.28}$$

Firstly, let us check that this action is real, namely $S^* = S$. In fact,

$$S^* = \int \mathrm{d}^4x \left[(\chi^\dagger \mathrm{i}\bar{\sigma}\cdot\partial\chi)^\dagger - \frac{im}{2}(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi)\right],$$

where the first term $(\chi^\dagger\mathrm{i}\bar{\sigma}\cdot\partial\chi)^\dagger = -\mathrm{i}(\partial\chi^\dagger)\mathrm{i}\bar{\sigma}\chi$ is identical to the original kinetic term upon integration by parts. Thus we see that $S^* = S$.

Now we vary the action with respect to $\chi^\dagger$, that gives

$$0 = \frac{\delta S}{\delta\chi^\dagger} = \mathrm{i}\bar{\sigma}\cdot\partial\chi - \frac{im}{2}\cdot 2\sigma^2\chi^* = 0, \tag{3.29}$$

which is exactly the Majorana equation.

(c) Let us rewrite the Dirac Lagrangian in terms of two-component spinors:

$$\begin{aligned}
\mathcal{L} &= \bar{\psi}(\mathrm{i}\partial\!\!\!/ - m)\psi \\
&= \begin{pmatrix}\chi_1^\dagger & -\mathrm{i}\chi_2^T\sigma^2\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}-m & \mathrm{i}\sigma^\mu\partial_\mu\\ \mathrm{i}\bar{\sigma}^\mu\partial_\mu & -m\end{pmatrix}\begin{pmatrix}\chi_1\\ \mathrm{i}\sigma^2\chi_2^*\end{pmatrix} \\
&= \mathrm{i}\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + \mathrm{i}\chi_2^T\bar{\sigma}^{\mu*}\partial_\mu\chi_2^* - \mathrm{i}m\big(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\big) \\
&= \mathrm{i}\chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 + \mathrm{i}\chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2 - \mathrm{i}m\big(\chi_2^T\sigma^2\chi_1 - \chi_1^\dagger\sigma^2\chi_2^*\big),
\end{aligned} \tag{3.30}$$

where the equality should be understood to hold up to a total derivative term.

(d) The familiar global $U(1)$ symmetry of the Dirac Lagrangian $\psi \to e^{\mathrm{i}\alpha}\psi$ now becomes $\chi_1 \to e^{\mathrm{i}\alpha}\chi_1$, $\chi_2 \to e^{-\mathrm{i}\alpha}\chi_2$. The associated Nöther current is

$$J^\mu = \bar{\psi}\gamma^\mu\psi = \chi_1^\dagger\bar{\sigma}^\mu\chi_1 - \chi_2^\dagger\bar{\sigma}^\mu\chi_2. \tag{3.31}$$

To show its divergence $\partial_\mu J^\mu$ vanishes, we make use of the equations of motion:

$$\begin{aligned}
&\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_1 - \mathrm{i}m\sigma^2\chi_2^* = 0,\\
&\mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_2 - \mathrm{i}m\sigma^2\chi_1^* = 0,\\
&\mathrm{i}(\partial_\mu\chi_1^\dagger)\bar{\sigma}^\mu - \mathrm{i}m\chi_2^T\sigma^2 = 0,\\
&\mathrm{i}(\partial_\mu\chi_2^\dagger)\bar{\sigma}^\mu - \mathrm{i}m\chi_1^T\sigma^2 = 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
\partial_\mu J^\mu &= (\partial_\mu\chi_1^\dagger)\bar{\sigma}^\mu\chi_1 + \chi_1^\dagger\bar{\sigma}^\mu\partial_\mu\chi_1 - (\partial_\mu\chi_2^\dagger)\bar{\sigma}^\mu\chi_2 - \chi_2^\dagger\bar{\sigma}^\mu\partial_\mu\chi_2\\
&= m\big(\chi_2^T\sigma^2\chi_1 + \chi_1^\dagger\sigma^2\chi_2^* - \chi_1^T\sigma^2\chi_2 - \chi_2^\dagger\sigma^2\chi_1^*\big) = 0.
\end{aligned} \tag{3.32}$$

In a similar way, one can also show that the Nöther currents associated with the global symmetries of Majorana fields have vanishing divergence.

(e) To quantize the Majorana theory, we introduce the canonical anticommutation relation,

$$\big\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\big\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

and also expand the Majorana field χ into modes. To motivate the mode expansion, we note that the Majorana Langrangian can be obtained by replacing the spinor χ_2 in the Dirac Lagrangian (3.30) with χ_1. Then, according to our experience in Dirac theory, it can be found that

$$\chi(x) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{p\cdot\sigma}{2E_p}}\sum_a\Big[\xi_a a_a(\mathbf{p})e^{-\mathrm{i}p\cdot x} + (-\mathrm{i}\sigma^2)\xi_a^* a_a^\dagger(\mathbf{p})e^{\mathrm{i}p\cdot x}\Big]. \tag{3.33}$$

Then with the canonical anticommutation relation above, we can find the anticommutators between annihilation and creation operators:

$$\{a_a(\mathbf{p}), a_b^\dagger(\mathbf{q})\} = \delta_{ab}\delta^{(3)}(\mathbf{p}-\mathbf{q}), \qquad \{a_a(\mathbf{p}), a_b(\mathbf{q})\} = \{a_a^\dagger(\mathbf{p}), a_b^\dagger(\mathbf{q})\} = 0. \tag{3.34}$$

On the other hand, the Hamiltonian of the theory can be obtained by Legendre transforming the Lagrangian:

$$H = \int\mathrm{d}^3x\left(\frac{\delta\mathcal{L}}{\delta\dot{\chi}}\dot{\chi} - \mathcal{L}\right) = \int\mathrm{d}^3x\left[\mathrm{i}\chi^\dagger\boldsymbol{\sigma}\cdot\nabla\chi + \frac{\mathrm{i}m}{2}\big(\chi^\dagger\sigma^2\chi^* - \chi^T\sigma^2\chi\big)\right]. \tag{3.35}$$

Then we can also represent the Hamiltonian H in terms of modes:

$$H = \int\mathrm{d}^3x\int\frac{\mathrm{d}^3p\mathrm{d}^3q}{(2\pi)^6\sqrt{2E_p2E_q}}\sum_{a,b}\bigg[\Big(\xi_a^\dagger a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^T(\mathrm{i}\sigma^2)a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)$$

$$
\begin{aligned}
&\times (\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}\Big(\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} - (-\mathrm{i}\sigma^2)\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\\
&+\frac{\mathrm{i}m}{2}\Big(\xi_a^\dagger a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^T(\mathrm{i}\sigma^2)a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\\
&\times (\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\Big(\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}} + (-\mathrm{i}\sigma^2)\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\\
&-\frac{\mathrm{i}m}{2}\Big(\xi_a^T a_a(\mathbf{p})e^{\mathrm{i}\mathbf{p}\cdot\mathbf{x}} + \xi_a^\dagger(\mathrm{i}\sigma^2)a_a^\dagger(\mathbf{p})e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}\Big)\\
&\times (\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big(\xi_b a_b(\mathbf{q})e^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} + (-\mathrm{i}\sigma^2)\xi_b^* a_b^\dagger(\mathbf{q})e^{-\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\Big)\Big]\\
=&\int \mathrm{d}^3x\int\frac{\mathrm{d}^3p\,\mathrm{d}^3q}{(2\pi)^6\sqrt{2E_p2E_q}}\sum_{a,b}\Big\{a_a^\dagger(\mathbf{p})a_b(\mathbf{q})\xi_a^\dagger\Big[(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}\\
&+\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*(-\mathrm{i}\sigma^2) - \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big]\xi_b e^{-\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\\
&+a_a^\dagger(\mathbf{p})a_b^\dagger(\mathbf{q})\xi_a^\dagger\Big[-(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^* e^{-\mathrm{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\\
&+a_a(\mathbf{p})a_b(q)\xi_a^T\Big[(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma} + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*(-\mathrm{i}\sigma^2)\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}\Big]\xi_b e^{\mathrm{i}(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\\
&+a_a(\mathbf{p})a_b^\dagger(\mathbf{q})\xi_a^T\Big[-(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{q}\cdot\boldsymbol{\sigma})\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{q\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{q\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^* e^{\mathrm{i}(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\Big\}\\
=&\int\frac{\mathrm{d}^3p}{(2\pi)^3 2E_p}\sum_{a,b}\Big\{a_a^\dagger(\mathbf{p})a_b(\mathbf{p})\xi_a^\dagger\Big[(\sqrt{p\cdot\sigma})^\dagger(-p\cdot\boldsymbol{\sigma})\sqrt{p\cdot\sigma}\\
&+\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{p\cdot\sigma})^*(-\mathrm{i}\sigma^2) - \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{p\cdot\sigma}\Big]\xi_b\\
&+a_a^\dagger(\mathbf{p})a_b^\dagger(-\mathbf{p})\xi_a^\dagger\Big[-(\sqrt{p\cdot\sigma})^\dagger(\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{\tilde{p}\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{\tilde{p}\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{\tilde{p}\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^*\\
&+a_a(\mathbf{p})a_b(-\mathbf{p})\xi_a^T\Big[(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{\tilde{p}\cdot\sigma} + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{\tilde{p}\cdot\sigma})^*(-\mathrm{i}\sigma^2)\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{\tilde{p}\cdot\sigma}\Big]\xi_b\\
&+a_a(\mathbf{p})a_b^\dagger(\mathbf{p})\xi_a^T\Big[-(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger(-\mathbf{p}\cdot\boldsymbol{\sigma})\sqrt{p\cdot\sigma}(-\mathrm{i}\sigma^2) + \frac{\mathrm{i}m}{2}(\mathrm{i}\sigma^2)(\sqrt{p\cdot\sigma})^\dagger\sigma^2(\sqrt{p\cdot\sigma})^*\\
&\quad-\frac{\mathrm{i}m}{2}(\sqrt{p\cdot\sigma})^T\sigma^2\sqrt{p\cdot\sigma}(-\mathrm{i}\sigma^2)\Big]\xi_b^*\Big\}
\end{aligned}
$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2E_p} \sum_{a,b} \frac{1}{2}\left(E_p^2 + |\mathbf{p}|^2 + m^2\right)\left[a_a^\dagger(\mathbf{p})a_b(\mathbf{p})\xi_a^\dagger\xi_b - a_a(\mathbf{p})a_b^\dagger(\mathbf{p})\xi_a^T\xi_b^*\right]$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{E_p}{2}\sum_a \left[a_a^\dagger(\mathbf{p})a_a(\mathbf{p}) - a_a(\mathbf{p})a_a^\dagger(\mathbf{p})\right]$$

$$= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_p \sum_a a_a^\dagger(\mathbf{p})a_a(\mathbf{p}). \tag{3.36}$$

In the calculation above, each step goes as follows in turn: (1) Substituting the mode expansion for χ into the Hamiltonian. (2) Collecting the terms into four groups, characterized by $a^\dagger a$, $a^\dagger a^\dagger$, aa and $aa^\dagger$. (3) Integrating over d^3x to produce a delta function, with which one can further finish the integration over d^3q. (4) Using the following relations to simplify the spinor matrices:

$$(p\cdot\sigma)^2 = (p\cdot\bar{\sigma})^2 = E_p^2 + |\mathbf{p}|^2, \qquad (p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2, \qquad \mathbf{p}\cdot\boldsymbol{\sigma} = \tfrac{1}{2}(p\cdot\bar{\sigma} - p\cdot\sigma).$$

In this step, the $a^\dagger a^\dagger$ and aa terms vanish, while the $aa^\dagger$ and $a^\dagger a$ terms remain. (5) Using the normalization $\xi_a^\dagger\xi_b = \delta_{ab}$ to eliminate spinors. (6) Using the anticommutator $\{a_a(\mathbf{p}), a_a^\dagger(\mathbf{p})\} = \delta^{(3)}(\mathbf{0})$ to further simplify the expression. In this step we have throw away a constant term $-\frac{1}{2}E_p\delta^{(3)}(\mathbf{0})$ in the integrand. The minus sign of this term indicates that the vacuum energy contributed by Majorana field is negative. With these steps done, we find the desired result, as shown above.

3.5 Supersymmetry

(a) In this problem we briefly study the Wess-Zumino model, which may be the simplest supersymmetric field theory in 4 dimensional spacetime. Firstly let us consider the massless case, in which the Lagrangian is given by

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^*F, \tag{3.37}$$

where ϕ is a complex scalar field, χ is a Weyl fermion, and F is a complex auxiliary scalar field. By auxiliary we mean a field with no kinetic term in the Lagrangian and thus it does not propagate, or equivalently, it has no particle excitation. However, in the following, we will see that it is crucial to maintain the off-shell supersymmtry of the theory.

The supersymmetry transformation in its infinitesimal form is given by:

$$\delta\phi = -i\epsilon^T\sigma^2\chi, \tag{3.38a}$$

$$\delta\chi = \epsilon F + \sigma^\mu(\partial_\mu\phi)\sigma^2\epsilon^*, \tag{3.38b}$$

$$\delta F = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\chi, \tag{3.38c}$$

where ϵ is a 2-component Grassmann variable. Now let us show that the Lagrangian is invariant (up to a total divergence) under this supersymmetric transformation. This can be checked term by term, as follows:

$$\delta(\partial_\mu\phi^*\partial^\mu\phi) = \mathrm{i}\big(\partial_\mu\chi^\dagger\sigma^2\epsilon^*\big)\partial^\mu\phi + (\partial_\mu\phi^*)\big(-\mathrm{i}\epsilon^T\sigma^2\partial^\mu\chi\big),$$

$$\begin{aligned}\delta(\chi^\dagger \mathrm{i}\bar\sigma^\mu\partial_\mu\chi) =& \big(F^*\epsilon^\dagger + \epsilon^T\sigma^2\sigma^\nu\partial_\nu\phi^*\big)\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\big(\epsilon\partial_\mu F + \sigma^\nu\sigma^2\epsilon^*\partial_\mu\partial_\nu\phi\big)\\ =& \,\mathrm{i}F^*\epsilon^\dagger\bar\sigma^2\partial_\mu\chi + \mathrm{i}\partial_\mu\big[\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\phi^*)\chi\big] - \mathrm{i}\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\partial_\mu\phi^*)\chi\\ &+ \mathrm{i}\chi^\dagger\bar\sigma^\mu\epsilon\partial_\mu F + \mathrm{i}\chi^\dagger\bar\sigma^\mu\sigma^\nu\sigma^2\epsilon^*\partial_\mu\partial_\nu\phi\\ =& \,\mathrm{i}F^*\epsilon^\dagger\bar\sigma^2\partial_\mu\chi + \mathrm{i}\partial_\mu\big[\epsilon^T\sigma^2\sigma^\nu\bar\sigma^\mu(\partial_\nu\phi^*)\chi\big] - \mathrm{i}\epsilon^T\sigma^2(\partial^2\phi^*)\chi\\ &+ \mathrm{i}\chi^\dagger\bar\sigma^\mu\epsilon\partial_\mu F + \mathrm{i}\chi^\dagger\sigma^2\epsilon^*\partial^2\phi,\\ \delta(F^*F) =& \,\mathrm{i}(\partial_\mu\chi^\dagger)\bar\sigma^\mu\epsilon F - \mathrm{i}F^*\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi,\end{aligned}$$

where we have used $\bar\sigma^\mu\sigma^\nu\partial_\mu\partial_\nu = \partial^2$. Now summing the three terms above, we get:

$$\delta\mathcal{L} = \mathrm{i}\partial_\mu\Big[\chi^\dagger\sigma^2\epsilon^*\partial^\mu\phi + \chi^\dagger\bar\sigma^\mu\epsilon F + \phi^*\epsilon^T\sigma^2\big(\sigma^\mu\sigma^\nu\sigma_\nu - \partial^\mu\big)\chi\Big], \tag{3.39}$$

which is indeed a total derivative.

(b) Now let us add the mass term in to the original massless Lagrangian:

$$\Delta\mathcal{L} = \big(m\phi F + \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2\chi\big) + \text{c.c.} \tag{3.40}$$

Let us show that this mass term is also invariant under the supersymmetry transformation, up to a total derivative:

$$\begin{aligned}\delta(\Delta\mathcal{L}) =& -\mathrm{i}m\epsilon^T\sigma^2\chi F - \mathrm{i}m\phi\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi + \tfrac{1}{2}\mathrm{i}m[\epsilon^T F + \epsilon^\dagger(\sigma^2)^T(\sigma^\mu)^T\partial_\mu\phi]\sigma^2\chi\\ &+ \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2[\epsilon F + \sigma^\mu(\partial_\mu\phi)\sigma^2\epsilon^*] + \text{c.c.}\\ =& -\tfrac{1}{2}\mathrm{i}mF(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon) - \mathrm{i}m\phi\epsilon^\dagger\bar\sigma^\mu\partial_\mu\chi\\ &- \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)\epsilon^\dagger\bar\sigma^\mu\chi + \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)\chi^T(\bar\sigma^\mu)^T\epsilon^* + \text{c.c.}\\ =& -\tfrac{1}{2}\mathrm{i}mF(\epsilon^T\sigma^2\chi - \chi^T\sigma^2\epsilon) - \mathrm{i}m\partial_\mu(\phi\epsilon^\dagger\bar\sigma^\mu\chi)\\ &+ \tfrac{1}{2}\mathrm{i}m(\partial_\mu\phi)[\epsilon^\dagger\bar\sigma^\mu\chi + \chi^T(\bar\sigma^\mu)^T\epsilon^*] + \text{c.c}\\ =& -\mathrm{i}m\partial_\mu(\phi\epsilon^\dagger\bar\sigma^\mu\chi) + \text{c.c}\end{aligned} \tag{3.41}$$

where we have used the following relations:

$$(\sigma^2)T = -\sigma^2, \qquad \sigma^2(\sigma^\mu)^T\sigma^2 = \bar\sigma^\mu, \qquad \epsilon^T\sigma^2\chi = \chi^T\sigma^2\epsilon, \qquad \epsilon^\dagger\bar\sigma^\mu\chi = -\chi^T(\bar\sigma^\mu)^T\epsilon^*.$$

Now let us write down the Lagrangian with the mass term:

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + F^*F + \big(m\phi F + \tfrac{1}{2}\mathrm{i}m\chi^T\sigma^2\chi + \text{c.c.}\big). \tag{3.42}$$

Varying the Lagrangian with respect to F^*, we get the corresponding equation of motion:

$$F = -m\phi^*. \tag{3.43}$$

Substitute this algebraic equation back into the Lagrangian to eliminate the field F, we get

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + \chi^\dagger\mathrm{i}\bar\sigma^\mu\partial_\mu\chi + \tfrac{1}{2}\big(\mathrm{i}m\chi^T\sigma^2\chi + \text{c.c.}\big). \tag{3.44}$$

Thus we see that the scalar field ϕ and the spinor field χ have the same mass.

(c) We can also include interactions into this model. Generally, we can write a Lagrangian with nontrivial interactions containing fields ϕ_i, χ_i and F_i $(i = 1, \cdots, n)$, as

$$\mathcal{L} = \partial_\mu\phi_i^*\partial^\mu\phi_i + \chi_i^\dagger \mathrm{i}\bar{\sigma}^\mu\partial_\mu\chi_i + F_i^*F_i + \left[F_i\frac{\partial W[\phi]}{\partial\phi_i} + \frac{\mathrm{i}}{2}\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + \text{c.c.}\right], \tag{3.45}$$

where $W[\phi]$ is an arbitrary function of ϕ_i.

To see this Lagrangian is supersymmetry invariant, we only need to check the interactions terms in the square bracket:

$$\begin{aligned}
&\delta\left[F_i\frac{\partial W[\phi]}{\partial\phi_i} + \frac{\mathrm{i}}{2}\frac{\partial^2 W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j + \text{c.c.}\right]\\
=&-\mathrm{i}\epsilon^\dagger\bar{\sigma}^\mu(\partial_\mu\chi_i)\frac{\partial W}{\partial\phi_i} + F_i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(-\mathrm{i}\epsilon^T\sigma^2\chi_j) + \frac{\mathrm{i}}{2}\frac{\partial^3 W}{\partial\phi_i\partial\phi_j\partial\phi_k}(-\mathrm{i}\epsilon^T\sigma^2\chi_k)\chi_i^T\sigma^2\chi_j\\
&+\frac{\mathrm{i}}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\Big[\big(\epsilon^T F_i + \epsilon^\dagger(\sigma^2)^T(\sigma^\mu)^T\partial_\mu\phi_i\big)\sigma^2\chi_j + \chi_i^T\sigma^2\big(\epsilon F_j + \sigma^\mu\partial_\mu\phi_j\sigma^2\epsilon^*\big)\Big] + \text{c.c.}.
\end{aligned}$$

The term proportional to $\partial^3 W/\partial\phi^3$ vanishes. To see this, we note that the partial derivatives with respect to ϕ_i are commutable, hence $\partial^3 W/\partial\phi_i\partial\phi_j\partial\phi_k$ is totally symmetric on i, j, k. However, we also have the following identity:

$$(\epsilon^T\sigma^2\chi_k)(\chi_i^T\sigma^2\chi_j) + (\epsilon^T\sigma^2\chi_i)(\chi_j^T\sigma^2\chi_k) + (\epsilon^T\sigma^2\chi_j)(\chi_k^T\sigma^2\chi_i) = 0, \tag{3.46}$$

which can be directly checked by brute force. Then it can be easily seen that the $\partial^3 W/\partial\phi^3$ term vanishes indeed. On the other hand, the terms containing F also sum to zero, which is also straightforward to justify. Hence the terms left now are

$$\begin{aligned}
&-\mathrm{i}\epsilon^\dagger\bar{\sigma}^\mu(\partial_\mu\chi_i)\frac{\partial W}{\partial\phi_i} + \mathrm{i}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\epsilon^\dagger(\sigma^2)^T(\sigma^\mu)^T(\partial_\mu\phi_i)\sigma^2\chi_j\\
=&-\mathrm{i}\partial_\mu\Big(\epsilon^\dagger\bar{\sigma}^\mu\chi_i\frac{\partial W}{\partial\phi_i}\Big) + \mathrm{i}\epsilon^\dagger\bar{\sigma}^\mu\chi_i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\partial_\mu\phi_j - \mathrm{i}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\epsilon^\dagger\bar{\sigma}^\mu(\partial_\mu\phi_i)\chi_j\\
=&-\mathrm{i}\partial_\mu\Big(\epsilon^\dagger\bar{\sigma}^\mu\chi_i\frac{\partial W}{\partial\phi_i}\Big),
\end{aligned} \tag{3.47}$$

which is a total derivative. Thus we conclude that the Lagrangian (3.45) is supersymmetrically invariant up to a total derivative.

Let us end up with a explicit example, in which we choose $n = 1$ and $W[\phi] = g\phi^3/3$. Then the Lagrangian (3.45) becomes

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + F^*F + \Big(gF\phi^2 + i\phi\chi^T\sigma^2\chi + \text{c.c.}\Big). \tag{3.48}$$

We can eliminate F by solving it from its field equation,

$$F + g(\phi^*)^2 = 0. \tag{3.49}$$

Substituting this back into the Lagrangian, we get

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - g^2(\phi^*\phi)^2 + ig(\phi\chi^T\sigma^2\chi - \phi^*\chi^\dagger\sigma^2\chi^*). \tag{3.50}$$

This is a Lagrangian of massless complex scalar and a Weyl spinor, with ϕ^4 and Yukawa interactions. The field equations can be easily got by the variation.

3.6 Fierz transformations

In this problem, we derive the generalized Fierz transformation, with which one can express $(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4)$ as a linear combination of $(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2)$, where Γ^A is any normalized Dirac matrices from the following set:

$$\{\ 1, \gamma^\mu, \sigma^{\mu\nu} = \tfrac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu], \gamma^5\gamma^\mu, \gamma^5 = -\mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3\ \}.$$

(a) The Dirac matrices Γ^A are normalized according to

$$\mathrm{tr}\,(\Gamma^A\Gamma^B) = 4\delta^{AB}. \tag{3.51}$$

For instance, the unit element 1 is already normalized, since $\mathrm{tr}\,(1\cdot 1) = 4$. For Dirac matrices containing one γ^μ, we calculate the trace in Weyl representation without loss of generality. Then the representation of

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

gives $\mathrm{tr}\,(\gamma^\mu\gamma^\mu) = 2\,\mathrm{tr}\,(\sigma^\mu\bar{\sigma}^\mu)$ (no sum on μ). For $\mu = 0$, we have $\mathrm{tr}\,(\gamma^0\gamma^0) = 2\,\mathrm{tr}\,(1_{2\times 2}) = 4$, and for $\mu = i = 1, 2, 3$, we have $\mathrm{tr}\,(\gamma^i\gamma^i) = -2\,\mathrm{tr}\,(\sigma^i\sigma^i) = -2\,\mathrm{tr}\,(1_{2\times 2}) = -4$ (no sum on i). Thus the normalized gamma matrices are γ^0 and $\mathrm{i}\gamma^i$.

In the same way, we can work out the rest of the normalized Dirac matrices, as:

$$\begin{aligned}
&\mathrm{tr}\,(\sigma^{0i}\sigma^{0i}) = -2\,\mathrm{tr}\,(\sigma^i\sigma^i) = -4, && \text{(no sum on } i)\\
&\mathrm{tr}\,(\sigma^{ij}\sigma^{ij}) = 2\,\mathrm{tr}\,(\sigma^k\sigma^k) = 4, && \text{(no sum on } i,j,k)\\
&\mathrm{tr}\,(\gamma^5\gamma^5) = 4, && \\
&\mathrm{tr}\,(\gamma^5\gamma^0\gamma^5\gamma^0) = -4, && \mathrm{tr}\,(\gamma^5\gamma^i\gamma^5\gamma^i) = 4.
\end{aligned}$$

Thus the 16 normalized elements are:

$$\{\ 1,\ \gamma^0,\ i\gamma^i,\ i\sigma^{0i},\ \sigma^{ij},\ \gamma^5,\ i\gamma^5\gamma^0,\ \gamma^5\gamma^i\ \}. \tag{3.52}$$

(b) Now we derive the desired Fierz identity, which can be written as:

$$(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{C,D} C^{AB}{}_{CD}(\bar{u}_1\Gamma^C u_4)(\bar{u}_3\Gamma^D u_2). \tag{3.53}$$

Left-multiplying the equality by $(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)$, we get:

$$(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \sum_{CD} C^{AB}{}_{CD}\,\mathrm{tr}\,(\Gamma^E\Gamma^C)\,\mathrm{tr}\,(\Gamma^F\Gamma^D). \tag{3.54}$$

The left hand side:

$$(\bar{u}_2\Gamma^F u_3)(\bar{u}_4\Gamma^E u_1)(\bar{u}_1\Gamma^A u_2)(\bar{u}_3\Gamma^B u_4) = \bar{u}_4\Gamma^E\Gamma^A\Gamma^F\Gamma^B u_4 = \mathrm{tr}\,(\Gamma^E\Gamma^A\Gamma^F\Gamma^B);$$

the right hand side:

$$\sum_{C,D} C^{AB}{}_{CD}\,\mathrm{tr}\,(\Gamma^E\Gamma^C)\,\mathrm{tr}\,(\Gamma^F\Gamma^D) = \sum_{C,D} C^{AB}{}_{CD}4\delta^{EC}4\delta^{FD} = 16C^{AB}{}_{EF},$$

thus we conclude:

$$C^{AB}{}_{CD} = \tfrac{1}{16}\,\mathrm{tr}\,(\Gamma^C\Gamma^A\Gamma^D\Gamma^B). \tag{3.55}$$

(c) Now we derive two Fierz identities as particular cases of the results above. The first one is:

$$(\bar{u}_1 u_2)(\bar{u}_3 u_4) = \sum_{C,D} \frac{\mathrm{tr}\,(\Gamma^C \Gamma^D)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^D u_2). \tag{3.56}$$

The traces on the right hand side do not vanish only when $C = D$, thus we get:

$$\begin{aligned}(\bar{u}_1 u_2)(\bar{u}_3 u_4) &= \sum_C \tfrac{1}{4} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^C u_2) \\ &= \tfrac{1}{4}\Big[(\bar{u}_1 u_4)(\bar{u}_3 u_2) + (\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2) + \tfrac{1}{2}(\bar{u}_1 \sigma^{\mu\nu} u_4)(\bar{u}_3 \sigma_{\mu\nu} u_2) \\ &\quad - (\bar{u}_1 \gamma^5 \gamma^\mu u_4)(\bar{u}_3 \gamma^5 \gamma_\mu u_2) + (\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2)\Big].\end{aligned} \tag{3.57}$$

The second example is:

$$(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4) = \sum_{C,D} \frac{\mathrm{tr}\,(\Gamma^C \gamma^\mu \Gamma^D \gamma_\mu)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^D u_2). \tag{3.58}$$

Again, the traces vanish if $\Gamma^C \gamma^\mu \neq C \propto \Gamma^D \gamma^\mu$ with C a commuting number, which implies that $\Gamma^C = \Gamma^D$. That is,

$$\begin{aligned}(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4) &= \sum_C \frac{\mathrm{tr}\,(\Gamma^C \gamma^\mu \Gamma^C \gamma_\mu)}{16} (\bar{u}_1 \Gamma^C u_4)(\bar{u}_3 \Gamma^C u_2) \\ &= \tfrac{1}{4}\Big[4(\bar{u}_1 u_4)(\bar{u}_3 u_2) - 2(\bar{u}_1 \gamma^\mu u_4)(\bar{u}_3 \gamma_\mu u_2) \\ &\quad - 2(\bar{u}_1 \gamma^5 \gamma^\mu u_4)(\bar{u}_3 \gamma^5 \gamma_\mu u_2) - 4(\bar{u}_1 \gamma^5 u_4)(\bar{u}_3 \gamma^5 u_2)\Big].\end{aligned} \tag{3.59}$$

We note that the normalization of Dirac matrices has been properly taken into account by raising or lowering of Lorentz indices.

3.7 The discrete symmetries P, C and T

(a) In this problem, we will work out the C, P and T transformations of the bilinear $\bar{\psi}\sigma^{\mu\nu}\psi$, with $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Firstly,

$$P\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})P = \tfrac{i}{2}\bar{\psi}(t,-\mathbf{x})\gamma^0[\gamma^\mu,\gamma^\nu]\gamma^0\psi(t,-\mathbf{x}).$$

With the relations $\gamma^0[\gamma^0,\gamma^i]\gamma^0 = -[\gamma^0,\gamma^i]$ and $\gamma^0[\gamma^i,\gamma^j]\gamma^0 = [\gamma^i,\gamma^j]$, we get:

$$P\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})P = \begin{cases} -\bar{\psi}(t,-\mathbf{x})\sigma^{0i}\psi(t,-\mathbf{x}); \\ \bar{\psi}(t,-\mathbf{x})\sigma^{ij}\psi(t,-\mathbf{x}). \end{cases} \tag{3.60}$$

Secondly,

$$T\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})T = -\tfrac{i}{2}\bar{\psi}(-t,\mathbf{x})(-\gamma^1\gamma^3)[\gamma^\mu,\gamma^\nu]^*(\gamma^1\gamma^3)\psi(-t,\mathbf{x}).$$

Note that gamma matrices keep invariant under transposition, except γ^2, which changes the sign. Thus we have:

$$T\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})T = \begin{cases} \bar{\psi}(-t,\mathbf{x})\sigma^{0i}\psi(-t,\mathbf{x}); \\ -\bar{\psi}(-t,\mathbf{x})\sigma^{ij}\psi(-t,\mathbf{x}). \end{cases} \tag{3.61}$$

Thirdly,

$$C\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})C = -\tfrac{\mathrm{i}}{2}(-\mathrm{i}\gamma^0\gamma^2\psi)^T\sigma^{\mu\nu}(-\mathrm{i}\bar{\psi}\gamma^0\gamma^2)^T = \bar{\psi}\gamma^0\gamma^2(\sigma^{\mu\nu})^T\gamma^0\gamma^2\psi.$$

Note that γ^0 and γ^2 are symmetric while γ^1 and γ^3 are antisymmetric, we have

$$C\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x})C = -\bar{\psi}(t,\mathbf{x})\sigma^{\mu\nu}\psi(t,\mathbf{x}). \tag{3.62}$$

(b) Now we work out the C, P and T transformation properties of a scalar field ϕ. Our starting point is

$$Pa_{\mathbf{p}}P = a_{-\mathbf{p}}, \qquad Ta_{\mathbf{p}}T = a_{-\mathbf{p}}, \qquad Ca_{\mathbf{p}}C = b_{\mathbf{p}}.$$

Then, for a complex scalar field

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{\mathbf{k}}e^{-\mathrm{i}k\cdot x} + b_{\mathbf{k}}^\dagger e^{\mathrm{i}k\cdot x}\Big], \tag{3.63}$$

we have

$$P\phi(t,\mathbf{x})P = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{-\mathbf{k}}e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + b_{-\mathbf{k}}^\dagger e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi(t,-\mathbf{x}). \tag{3.64a}$$

$$T\phi(t,\mathbf{x})T = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[a_{-\mathbf{k}}e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + b_{-\mathbf{k}}^\dagger e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi(-t,\mathbf{x}). \tag{3.64b}$$

$$C\phi(t,\mathbf{x})C = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{\sqrt{2k^0}}\Big[b_{\mathbf{k}}e^{-\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})} + a_{\mathbf{k}}^\dagger e^{\mathrm{i}(k^0t-\mathbf{k}\cdot\mathbf{x})}\Big] = \phi^*(t,\mathbf{x}). \tag{3.64c}$$

As a consequence, we can deduce the C, P, and T transformation properties of the current $J^\mu = \mathrm{i}\big(\phi^*\partial^\mu\phi - (\partial^\mu\phi^*)\phi\big)$, as follows:

$$\begin{aligned} PJ^\mu(t,\mathbf{x})P &= (-1)^{s(\mu)}\mathrm{i}\big[\phi^*(t,-\mathbf{x})\partial^\mu\phi(t,-\mathbf{x}) - \big(\partial^\mu\phi^*(t,-\mathbf{x})\big)\phi(t,-\mathbf{x})\big] \\ &= (-1)^{s(\mu)}J^\mu(t,-\mathbf{x}), \end{aligned} \tag{3.65a}$$

where $s(\mu)$ is the label for space-time indices that equals to 0 when $\mu = 0$ and 1 when $\mu = 1,2,3$. In the similar way, we have

$$TJ^\mu(t,\mathbf{x})T = (-1)^{s(\mu)}J^\mu(-t,\mathbf{x}); \tag{3.65b}$$

$$CJ^\mu(t,\mathbf{x})C = -J^\mu(t,\mathbf{x}). \tag{3.65c}$$

One should be careful when playing with T — it is antihermitian rather than hermitian, and anticommutes, rather than commutes, with $\sqrt{-1}$.

(c) Any Lorentz-scalar hermitian local operator $\mathcal{O}(x)$ constructed from $\psi(x)$ and $\phi(x)$ can be decomposed into groups, each of which is a Lorentz-tensor hermitian operator and contains either $\psi(x)$ or $\phi(x)$ only. Thus to prove that $\mathcal{O}(x)$ is an operator of $CPT = +1$, it is enough to show that all Lorentz-tensor hermitian operators constructed from either $\psi(x)$ or $\phi(x)$ have correct CPT value. For operators constructed from $\psi(x)$, this has been done as listed in Table on Page 71 of Peskin & Schroeder; and for operators constructed from $\phi(x)$, we note that all such operators can be decomposed further into a product (including Lorentz inner product) of operators of the form

$$(\partial_{\mu_1}\cdots\partial_{\mu_m}\phi^\dagger)(\partial_{\mu_1}\cdots\partial_{\mu_n}\phi)+\text{c.c}$$

together with the metric tensor $\eta^{\mu\nu}$. But it is easy to show that any operator of this form has the correct CPT value, namely, has the same CPT value as a Lorentz tensor of rank $(m+n)$. Therefore we conclude that any Lorentz-scalar hermitian local operator constructed from ψ and ϕ has $CPT = +1$.

3.8 Bound states

(a) A positronium bound state with orbital angular momentum L and total spin S can be build by linear superposition of an electron state and a positron state, with the spatial wave function $\Psi_L(\mathbf{k})$ as the amplitude. Symbolically we have

$$|L,S\rangle \sim \sum_k \Psi_L(\mathbf{k})a^\dagger(\mathbf{k},s)b^\dagger(-\mathbf{k},s')|0\rangle.$$

Then, apply the space-inversion operator P, we get

$$P|L,S\rangle = \sum_k \Psi_L(-\mathbf{k})\eta_a\eta_b a^\dagger(-\mathbf{k},s)b^\dagger(\mathbf{k},s')|0\rangle = (-1)^L\eta_a\eta_b\sum_k \Psi_L(\mathbf{k})a^\dagger(\mathbf{k},s)b^\dagger(\mathbf{k},s')|0\rangle. \tag{3.66}$$

Note that $\eta_b = -\eta_a^*$, we conclude that $P|L,S\rangle = (-)^{L+1}|L,S\rangle$. Similarly,

$$C|L,S\rangle = \sum_k \Psi_L(\mathbf{k})b^\dagger(\mathbf{k},s)a^\dagger(-\mathbf{k},s')|0\rangle = (-1)^{L+S}\sum_k \Psi_L(\mathbf{k})b^\dagger(-\mathbf{k},s')a^\dagger(\mathbf{k},s)|0\rangle. \tag{3.67}$$

That is, $C|L,S\rangle = (-1)^{L+S}|L,S\rangle$. Then its easy to find the P and C eigenvalues of various states, listed as follows:

${}^SL$	${}^1S$	${}^3S$	${}^1P$	${}^3P$	${}^1D$	${}^3D$
P	$-$	$-$	$+$	$+$	$-$	$-$
C	$+$	$-$	$-$	$+$	$+$	$-$

(b) We know that a photon has parity eigenvalue -1 and C-eigenvalue -1. Thus we see that the decay into 2 photons are allowed for ${}^1S$ state but forbidden for ${}^3S$ state due to C-violation. That is, ${}^3S$ has to decay into at least 3 photons.

Chapter 4

Interacting Fields and Feynman Diagrams

4.1 Scalar field with a classical source

In this problem we consider the theory with the following Hamiltonian:

$$H = H_0 - \int \mathrm{d}^3\, j(t,\mathbf{x})\phi(x), \tag{4.1}$$

where H_0 is the Hamiltonian for free Klein-Gordon field ϕ, and j is a classical source.

(a) We calculate the probability that the source creates no particles. The corresponding amplitude is given by the inner product between the in-state and the out-state, both of which are vacuum in our case. Therefore,

$$\begin{aligned} P(0) =& \big|{}_{\text{out}}\langle 0|0\rangle_{\text{in}}\big|^2 = \lim_{t\to(1-i\epsilon)\infty} \big|\langle 0|e^{-i2Ht}|0\rangle\big|^2 \\ =& \Big|\langle 0|T\exp\Big\{-i\int \mathrm{d}^4x\,\mathcal{H}_{\text{int}}\Big\}|0\rangle\Big|^2 = \Big|\langle 0|T\exp\Big\{i\int \mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle\Big|^2. \end{aligned} \tag{4.2}$$

(b) Now we expand this probability $P(0)$ to j^2. The amplitude reads,

$$\begin{aligned} \langle 0|T\exp\Big\{i\int \mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle =& 1 - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, j(x)\langle 0|T\phi_I(x)\phi_I(y)|0\rangle j(y) + O(j^4) \\ =& 1 - \frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4y\, j(x)j(y)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} + O(j^4) \\ =& 1 - \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2 + O(j^4). \end{aligned} \tag{4.3}$$

Thus the probability is given by,

$$P(0) = |1 - \frac{1}{2}\lambda + O(j^4)|^2 = 1 - \lambda + O(j^4), \tag{4.4}$$

where

$$\lambda \equiv \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2. \tag{4.5}$$

(c) We can calculate the probability $P(0)$ exactly, by working out the j^{2n} term of the expansion as,

$$\begin{aligned}
&\frac{i^{2n}}{(2n)!}\int \mathrm{d}^4x_1\cdots \mathrm{d}^4x_{2n}\, j(x_1)\cdots j(x_{2n})\langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle \\
=&\frac{i^{2n}(2n-1)(2n-3)\cdots 3\cdot 1}{(2n)!}\int \mathrm{d}^4x_1\cdots \mathrm{d}^4x_{2n}\, j(x_1)\cdots j(x_{2n}) \\
&\int \frac{\mathrm{d}^3p_1\cdots \mathrm{d}^3p_n}{(2\pi)^{3n}}\frac{1}{2^nE_{\mathbf{p}_1}\cdots E_{\mathbf{p}_n}}e^{i\mathbf{p}_1\cdot(\mathbf{x}_1-\mathbf{x}_2)}\cdots e^{i\mathbf{p}_n\cdot(\mathbf{x}_{2n-1}-\mathbf{x}_{2n})} \\
=&\frac{(-1)^n}{2^nn!}\Big(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}}\Big)^n=\frac{(-\lambda/2)^2}{n!}.
\end{aligned} \tag{4.6}$$

Then,

$$P(0)=\Big(\sum_{n=0}^{\infty}\frac{(-\lambda/2)^n}{n!}\Big)^2=e^{-\lambda}. \tag{4.7}$$

(d) Now we calculate the probability that the source creates one particle with momentum $\mathbf{k}$, which is given by,

$$P(k)=\Big|\langle\mathbf{k}|T\exp\Big\{i\int\mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle\Big|^2 \tag{4.8}$$

Expanding the amplitude to the first order in j, we get:

$$\begin{aligned}
P(k)=&\Big|\langle\mathbf{k}|0\rangle+i\int\mathrm{d}^4x\, j(x)\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{e^{ip\cdot x}}{\sqrt{2E_{\mathbf{p}}}}\langle\mathbf{k}|a_{\mathbf{p}}^\dagger|0\rangle+O(j^2)\Big|^2 \\
=&\Big|i\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\tilde{j}(p)}{\sqrt{2E_{\mathbf{p}}}}\sqrt{2E_{\mathbf{p}}}(2\pi)^3\delta(\mathbf{p}-\mathbf{k})\Big|^2=|\tilde{j}(k)|^2+O(j^3).
\end{aligned} \tag{4.9}$$

If we go on to work out all the terms, we get,

$$P(k)=\Big|\sum_n\frac{i(2n+1)(2n+1)(2n-1)\cdots 3\cdot 1}{(2n+1)!}\tilde{j}^{n+1}(k)\Big|^2=|\tilde{j}(k)|^2e^{-|\tilde{j}(k)|}. \tag{4.10}$$

(e) To calculate the probability that the source creates n particles, we write down the relevant amplitude,

$$\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\langle\mathbf{k}_1\cdots\mathbf{k}_n|T\exp\Big\{i\int\mathrm{d}^4x\, j(x)\phi_I(x)\Big\}|0\rangle. \tag{4.11}$$

Expanding this amplitude in terms of j, we find that the first nonvanishing term is the one of n'th order in j. Repeat the similar calculations above, we can find that the amplitude is:

$$\begin{aligned}
&\frac{i^n}{n!}\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\int\mathrm{d}^4x_1\cdots\mathrm{d}^4x_n\, j_1\cdots j_n\langle\mathbf{k}_1\cdots\mathbf{k}_n|\phi_1\cdots\phi_n|0\rangle+O(j^{n+2}) \\
=&\frac{i^n}{n!}\int\frac{\mathrm{d}^3k_1\cdots\mathrm{d}^3k_n\,\tilde{j}^n(k)}{(2\pi)^{3n}\sqrt{2^nE_{\mathbf{k}_1}\cdots E_{\mathbf{k}_n}}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^nn!}\Big(\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{|\tilde{j}(p)|^2}{2E_{\mathbf{p}}}\Big)^n.
\end{aligned} \tag{4.12}$$

Then we see the probability is given by,

$$P(n) = \frac{\lambda^n}{n!}e^{-\lambda}, \tag{4.13}$$

which is a Poisson distribution.

(f) It's easy to check that the Poisson distribution $P(n)$ satisfies the following identities:

$$\sum_n P(n) = 1. \tag{4.14}$$

$$\langle N \rangle = \sum_n nP(n) = \lambda. \tag{4.15}$$

The first one is almost trivial, and the second one can be obtained by acting $\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$ to both sides of the first identity. If we apply $\lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}$ again to the second identity, we get:

$$\langle (N - \langle N \rangle)^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2 = \lambda. \tag{4.16}$$

4.2 Decay of a scalar particle

This problem is based on the following Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi. \tag{4.17}$$

When $M > 2m$, a Φ particle can decay into two ϕ particles. We want to calculate the lifetime of the Φ particle to lowest order in μ.

The two-body decay rate is given in (4.86) of P&S,

$$\int \mathrm{d}\Gamma = \frac{1}{2M}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6}\frac{1}{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}\big|\mathcal{M}(\Phi(0) \to \phi(p_1)\phi(p_2))\big|^2(2\pi)^4\delta^{(4)}(p_\Phi - p_1 - p_2). \tag{4.18}$$

To lowest order in μ, the amplitude $\mathcal{M}$ is given by,

$$i\mathcal{M} = -2i\mu. \tag{4.19}$$

The delta function in our case reads,

$$\delta^{(4)}(p_\Phi - p_1 - p_2) = \delta(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2})\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2), \tag{4.20}$$

thus,

$$\Gamma = \frac{1}{2}\cdot\frac{2\mu^2}{M}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6}\frac{1}{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}(2\pi)^4\delta(M - E_{\mathbf{p}_1} - E_{\mathbf{p}_2})\delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2), \tag{4.21}$$

where an additional factor of 1/2 takes account of two identical ϕ's in final state. Furthermore, there are two mass-shell constraints,

$$m^2 + \mathbf{p}_i^2 = E_{\mathbf{p}_i}^2. \qquad (i = 1, 2) \tag{4.22}$$

Hence,

$$\Gamma = \frac{\mu^2}{M}\int \frac{\mathrm{d}^3 p_1}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}_1}^2}(2\pi)\delta(M-2E_{\mathbf{p}_1}) = \frac{\mu^2}{8\pi M}\Big(1-\frac{4m^2}{M^2}\Big)^{1/2}. \tag{4.23}$$

Then the lifetime τ of Φ is,

$$\tau = \Gamma^{-1} = \frac{8\pi M}{\mu^2}\Big(1-\frac{4m^2}{M^2}\Big)^{-1/2}. \tag{4.24}$$

4.3 Linear sigma model

In this problem, we study the linear sigma model described by the following Lagrangian,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\Phi^i\partial^\mu\Phi^i - \tfrac{1}{2}m^2\Phi^i\Phi^i - \tfrac{1}{4}\lambda(\Phi^i\Phi^i)^2. \tag{4.25}$$

Where Φ is an N-component scalar.

(a) We firstly compute the differential cross sections to the leading order in λ for the following three processes,

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \qquad \Phi^1\Phi^1 \to \Phi^1\Phi^1. \tag{4.26}$$

Since the masses of all incoming and outgoing particles are identical, the cross section is simply given by

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2 s}, \tag{4.27}$$

where s is the square of center-of-mass energy, and $\mathcal{M}$ is the scattering amplitude. From the Feynman rules it's easy to get,

$$\begin{aligned}&\mathcal{M}(\Phi^1\Phi^2 \to \Phi^1\Phi^2) = \mathcal{M}(\Phi^1\Phi^1 \to \Phi^2\Phi^2) = -2\mathrm{i}\lambda,\\ &\mathcal{M}(\Phi^1\Phi^1 \to \Phi^1\Phi^1) = -6\mathrm{i}\lambda.\end{aligned} \tag{4.28}$$

It follows immediately that

$$\begin{aligned}&\sigma(\Phi^1\Phi^2 \to \Phi^1\Phi^2) = \sigma M(\Phi^1\Phi^1 \to \Phi^2\Phi^2) = \frac{\lambda^2}{16\pi^2 s},\\ &\sigma(\Phi^1\Phi^1 \to \Phi^1\Phi^1) = \frac{9\lambda^2}{16\pi^2 s}.\end{aligned} \tag{4.29}$$

(b) Now we study the symmetry broken case, that is, $m^2 = -\mu^2 < 0$. Then, the scalar multiplet Φ can be parameterized as

$$\Phi = (\pi^1, \cdots, \pi^{N-1}, \sigma + v)^T, \tag{4.30}$$

where v is the VEV of $|\Phi|$, and equals to $\sqrt{\mu^2/\lambda}$ at tree level.

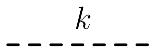
Substitute this into the Lagrangian, we get

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\pi^k)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\sigma\pi^k\pi^k$$

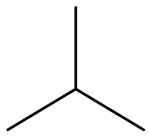
$$- \tfrac{\lambda}{4}\sigma^4 - \tfrac{\lambda}{2}\sigma^2(\pi^k\pi^k) - \tfrac{\lambda}{4}(\pi^k\pi^k)^2. \tag{4.31}$$

Then it's easy to read the Feynman rules from this expression:

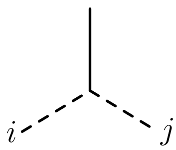
$$= \frac{\mathrm{i}}{k^2 - 2\mu^2}; \tag{4.32a}$$



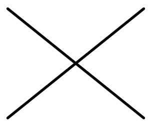
$$= \frac{\mathrm{i}\delta^{ij}}{k^2}; \tag{4.32b}$$



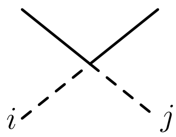
$$= 6\mathrm{i}\lambda v; \tag{4.32c}$$



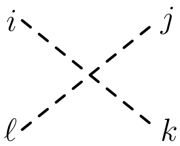
$$= -2\mathrm{i}\lambda v\delta^{ij}; \tag{4.32d}$$



$$= -6\mathrm{i}\lambda; \tag{4.32e}$$



$$= -2\mathrm{i}\lambda\delta^{ij}; \tag{4.32f}$$



$$= -2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}). \tag{4.32g}$$

(c) With the Feynman rules derived in (b), we can compute the amplitude

$$\mathcal{M}\big[\pi^i(p_1)\pi^j(p_2) \to \pi^k(p_3)\pi^\ell(p_4)\big],$$

as:

$$\begin{aligned}\mathcal{M} = (-2\mathrm{i}\lambda v)^2\Big[&\frac{\mathrm{i}}{s-2\mu^2}\delta^{ij}\delta^{k\ell} + \frac{\mathrm{i}}{t-2\mu^2}\delta^{ik}\delta^{j\ell} + \frac{\mathrm{i}}{u-2\mu^2}\delta^{i\ell}\delta^{jk}\Big]\\ &- 2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}),\end{aligned} \tag{4.33}$$

where s,t,u are Mandelstam variables (See Section 5.4 of P&S). Then, at the threshold $p_i = 0$, we have $s = t = u = 0$, and $\mathcal{M}$ vanishes.

On the other hand, if $N = 2$, then there is only one component in π, thus the amplitude reduces to

$$\mathcal{M} = -2\mathrm{i}\lambda\Big[\frac{2\mu^2}{s-2\mu^2} + \frac{2\mu^2}{t-2\mu^2} + \frac{2\mu^2}{u-2\mu^2} + 3\Big]$$

$$= 2\mathrm{i}\lambda\Big[\frac{s+t+u}{2\mu^2} + \mathcal{O}(p^4)\Big]. \tag{4.34}$$

In the second line we perform the Taylor expansion on s, t and u, which are of order $\mathcal{O}(p^2)$. Note that $s + t + u = 4m_\pi^2 = 0$, thus we see that $\mathcal{O}(p^2)$ terms are also canceled out.

(d) We minimize the potential with a small symmetry breaking term:

$$V = -\mu^2\Phi^i\Phi^i + \tfrac{\lambda}{4}(\Phi^i\Phi^i)^2 - a\Phi^N, \tag{4.35}$$

which yields the following equation that determines the VEV:

$$\big(-\mu^2 + \lambda\Phi^i\Phi^i\big)\Phi^i = a\delta^{iN}. \tag{4.36}$$

Thus, up to linear order in a, the VEV $\langle\Phi^i\rangle = (0, \cdots, 0, v)$ is

$$v = \sqrt{\frac{\mu^2}{\lambda}} + \frac{a}{2\mu^2}. \tag{4.37}$$

Now we repeat the derivation in (b) with this new VEV, and write the Lagrangian in terms of new field variable π^i and σ, as

$$\begin{aligned}\mathcal{L} = {}& \tfrac{1}{2}(\partial_\mu\pi^k)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - \tfrac{1}{2}\tfrac{\sqrt{\lambda}a}{\mu}\pi^k\pi^k - \tfrac{1}{2}(2\mu^2)\sigma^2 \\ & - \lambda v\sigma^3 - \lambda v\sigma\pi^k\pi^k - \tfrac{1}{4}\lambda\sigma^4 - \tfrac{\lambda}{2}\sigma^2(\pi^k\pi^k) - \tfrac{\lambda}{4}(\pi^k\pi^k)^2.\end{aligned} \tag{4.38}$$

The $\pi^i\pi^j \to \pi^k\pi^\ell$ amplitude is still given by

$$\begin{aligned}\mathcal{M} = {}& (-2\mathrm{i}\lambda v)^2\Big[\frac{\mathrm{i}}{s-2\mu^2}\delta^{ij}\delta^{k\ell} + \frac{\mathrm{i}}{t-2\mu^2}\delta^{ik}\delta^{j\ell} + \frac{\mathrm{i}}{u-2\mu^2}\delta^{i\ell}\delta^{jk}\Big] \\ & - 2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}).\end{aligned} \tag{4.39}$$

However this amplitude does not vanish at the threshold. Since the vertices $\lambda\nu \neq \sqrt{\lambda}\mu$ exactly even at tree level, and also s, t and u are not exactly zero in this case due to nonzero mass of π^i. Both deviations are proportional to a, thus we conclude that the amplitude $\mathcal{M}$ is also proportional to a.

4.4 Rutherford scattering

The Rutherford scattering is the scattering of an election by the coulomb field of a nucleus. In this problem, we calculate the cross section by treating the electromagnetic field as fixed classical background given by potential $A_\mu(x)$. Then the interaction Hamiltonian is,

$$H_I = \int \mathrm{d}^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{4.40}$$

(a) We first calculate the T-matrix to lowest order,

$$\begin{aligned}
{}_{\text{out}}\langle p'|p\rangle_{\text{in}} =&\langle p'|T\exp(-i\int \mathrm{d}^4x\, H_I)|p\rangle = \langle p'|p\rangle - ie\int \mathrm{d}^4x\, A_\mu(x)\langle p'|\bar{\psi}\gamma^\mu\psi|p\rangle + O(e^2)\\
=&\langle p'|p\rangle - ie\int \mathrm{d}^4x\, A_\mu(x)\bar{u}(p')\gamma^\mu u(p)e^{i(p'-p)\cdot x} + O(e^2)\\
=&(2\pi)^4\delta^{(4)}(p-p') - ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p) + O(e^2).
\end{aligned} \tag{4.41}$$

On the other hand,

$$ {}_{\text{out}}\langle p'|p\rangle_{\text{in}} = \langle p'|S|p\rangle = \langle p'|p\rangle + \langle p'|iT|p\rangle. \tag{4.42}$$

Thus to the first order of e, we get,

$$\langle p'|iT|p\rangle = -ie\bar{u}(p')\gamma^\mu u(p)\tilde{A}_\mu(p'-p). \tag{4.43}$$

(b) Now we calculate the cross section $\mathrm{d}\sigma$ in terms of the matrix elements $i\mathcal{M}$.

The incident wave packet $|\psi\rangle$ is defined to be:

$$|\psi\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{-i\mathbf{b}\cdot\mathbf{k}}}{\sqrt{2E_\mathbf{k}}}\psi(\mathbf{k})|\mathbf{k}\rangle, \tag{4.44}$$

where $\mathbf{b}$ is the impact parameter.

The probability that a scattered electron will be found within an infinitesimal element d^3p centered at $\mathbf{p}$ is,

$$\begin{aligned}
P =&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\Big|{}_{\text{out}}\langle\mathbf{p}|\psi\rangle_{\text{in}}\Big|^2\\
=&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_\mathbf{k}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')\Big({}_{\text{out}}\langle\mathbf{p}|\mathbf{k}\rangle_{\text{in}}\Big)\Big({}_{\text{out}}\langle\mathbf{p}|\mathbf{k}'\rangle_{\text{in}}\Big)^*e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}\\
=&\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_\mathbf{k}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')\Big(\langle\mathbf{p}|iT|\mathbf{k}\rangle\Big)\Big(\langle\mathbf{p}|iT|\mathbf{k}'\rangle\Big)^*e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}.
\end{aligned} \tag{4.45}$$

In the last equality we throw away the trivial scattering part from the S-matrix. Note that,

$$\langle p'|iT|p\rangle = i\mathcal{M}(2\pi)\delta(E_{\mathbf{p}'}-E_\mathbf{p}), \tag{4.46}$$

so we have,

$$P = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6\sqrt{2E_\mathbf{k}2E_{\mathbf{k}'}}}\psi(\mathbf{k})\psi^*(\mathbf{k}')|i\mathcal{M}|^2(2\pi)^2\delta(E_\mathbf{p}-E_\mathbf{k})\delta(E_\mathbf{p}-E_{\mathbf{k}'})e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')}. \tag{4.47}$$

The cross section $\mathrm{d}\sigma$ is given by:

$$\mathrm{d}\sigma = \int \mathrm{d}^2b\, P(\mathbf{b}), \tag{4.48}$$

thus the integration over $\mathbf{b}$ gives a delta function:

$$\int \mathrm{d}^2b\, e^{-i\mathbf{b}\cdot(\mathbf{k}-\mathbf{k}')} = (2\pi)^2\delta^{(2)}(\mathbf{k}_\perp-\mathbf{k}'_\perp). \tag{4.49}$$

The other two delta functions in the integrand can be modified as follows,

$$\delta(E_{\mathbf{k}} - E_{\mathbf{k}'}) = \frac{E_{\mathbf{k}}}{k_\parallel}\delta(k_\parallel - k'_\parallel) = \frac{1}{v}\delta(k_\parallel - k'_\parallel), \tag{4.50}$$

where we have used $|\mathbf{v}| = v = v_\parallel$. Taking all these delta functions into account, we get,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2E_{\mathbf{k}}}\frac{1}{v}\psi(\mathbf{k})\psi^*(\mathbf{k})|i\mathcal{M}|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}}). \tag{4.51}$$

Since the momentum of the wave packet should be localized around its central value, we can pull out the quantities involving energy $E_{\mathbf{k}}$ outside the integral,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}(2\pi)|\mathcal{M}|^2\delta(E_{\mathbf{p}} - E_{\mathbf{k}})\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})\psi^*(\mathbf{k}). \tag{4.52}$$

Recall the normalization of the wave packet,

$$\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})^*\psi(\mathbf{k}) = 1, \tag{4.53}$$

then,

$$\mathrm{d}\sigma = \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k \to p)|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}}). \tag{4.54}$$

We can further integrate over $|\mathbf{p}|$ to get the differential cross section $\mathrm{d}\sigma/\mathrm{d}\Omega$,

$$\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \int\frac{\mathrm{d}p\,p^2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k \to p)|^2(2\pi)\delta(E_{\mathbf{p}} - E_{\mathbf{k}}) \\
&= \int\frac{\mathrm{d}p\,p^2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{k}}}\frac{1}{v}|\mathcal{M}(k \to p)|^2(2\pi)\frac{E_{\mathbf{k}}}{k}\delta(p - k) \\
&= \frac{1}{(4\pi)^2}|\mathcal{M}(k,\theta)|^2.
\end{aligned} \tag{4.55}$$

In the last line we work out the integral by virtue of delta function, which constrains the outgoing momentum $|\mathbf{p}| = |\mathbf{k}|$ but leave the angle θ between $\mathbf{p}$ and $\mathbf{k}$ arbitrary. Thus the amplitude $\mathcal{M}(k,\theta)$ is a function of momentum $|\mathbf{k}|$ and angle θ.

(c) We work directly for the relativistic case. Firstly the Coulomb potential $A^0 = Ze/4\pi r$ in momentum space is

$$A^0(\mathbf{q}) = \frac{Ze}{|\mathbf{q}|^2}. \tag{4.56}$$

This can be easily worked out by Fourier transformation, with a "regulator" e^{-mr} inserted:

$$A^0(\mathbf{q}, m) \equiv \int \mathrm{d}^3x\, e^{-\mathrm{i}\mathbf{p}\cdot\mathbf{x}}e^{-mr}\frac{Ze}{4\pi r} = \frac{Ze}{|\mathbf{q}|^2 + m^2}. \tag{4.57}$$

This is simply Yukawa potential, and Coulomb potential is a limiting case when $m \to 0$.

The amplitude is given by

$$\mathrm{i}\mathcal{M}(k,\theta) = \mathrm{i}e\bar{u}(p)\gamma^\mu\tilde{A}_\mu(\mathbf{q})u(p) \qquad \text{with } \mathbf{q} = \mathbf{p} - \mathbf{k}. \tag{4.58}$$

Then we have the squared amplitude with initial spin averaged and final spin summed (See §5.1 of P&S for details), as,

$$\begin{aligned}\frac{1}{2}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}(k,\theta)|^2 =&\frac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\sum_{\text{spin}}\bar{u}(p)\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(p)\\ =&\frac{1}{2}e^2\tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q})\,\mathrm{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right]\\ =&2e^2\big[2(p\cdot\tilde{A})(k\cdot\tilde{A})+\big(m^2-(k\cdot p)\big)\tilde{A}^2\big].\end{aligned} \tag{4.59}$$

Note that

$$\tilde{A}^0(\mathbf{q})=\frac{Ze}{|\mathbf{p}-\mathbf{k}|^2}=\frac{Ze}{4|\mathbf{k}|^2\sin^2(\theta/2)}, \tag{4.60}$$

thus

$$\frac{1}{2}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}(k,\theta)|^2=\frac{Z^2e^4\big(1-v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^4v^2\sin^4(\theta/2)}, \tag{4.61}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{Z^2\alpha^2\big(1-v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^2v^2\sin^4(\theta/2)} \tag{4.62}$$

. In non-relativistic case, this formula reduces to

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{Z^2\alpha^2}{4m^2v^4\sin^4(\theta/2)} \tag{4.63}$$

Chapter 5

Elementary Processes of Quantum Electrodynamics

5.1 Coulomb scattering

In this problem we continue our study of the Coulomb scattering in Problem 4.4. Here we consider the relativistic case. Let's first recall some main points considered before. The Coulomb potential $A^0 = Ze/4\pi r$ in momentum space is

$$A^0(\mathbf{q}) = \frac{Ze}{|\mathbf{q}|^2}. \tag{5.1}$$

Then the scattering amplitude is given by

$$\mathrm{i}\mathcal{M}(k,\theta) = \mathrm{i}e\bar{u}(p)\gamma^\mu \tilde{A}_\mu(\mathbf{q})u(p) \qquad \text{with } \mathbf{q} = \mathbf{p} - \mathbf{k}. \tag{5.2}$$

Then we can derive the squared amplitude with initial spin averaged and final spin summed, as:

$$\begin{aligned} \frac{1}{2}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}(k,\theta)|^2 &= \frac{1}{2}e^2 \tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q}) \sum_{\text{spin}} \bar{u}(p)\gamma^\mu u(k)\bar{u}(k)\gamma^\nu u(p) \\ &= \frac{1}{2}e^2 \tilde{A}_\mu(\mathbf{q})\tilde{A}_\nu(\mathbf{q}) \,\mathrm{tr}\left[\gamma^\mu(\not{p}+m)\gamma^\nu(\not{k}+m)\right] \\ &= 2e^2\Big[2(p\cdot\tilde{A})(k\cdot\tilde{A}) + \big(m^2 - (k\cdot p)\big)\tilde{A}^2\Big]. \end{aligned} \tag{5.3}$$

Note that

$$\tilde{A}^0(\mathbf{q}) = \frac{Ze}{|\mathbf{p}-\mathbf{k}|^2} = \frac{Ze}{4|\mathbf{k}|^2\sin^2(\theta/2)}, \tag{5.4}$$

thus

$$\frac{1}{2}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}(k,\theta)|^2 = \frac{Z^2e^4\big(1 - v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^4 v^2 \sin^4(\theta/2)}, \tag{5.5}$$

Now, from the result of Problem 4.4(b), we know that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2}\Big(\tfrac{1}{2}\sum_{\text{spin}} |\mathcal{M}(k,\theta)|^2\Big) = \frac{Z^2\alpha^2\big(1 - v^2\sin^2\frac{\theta}{2}\big)}{4|\mathbf{k}|^2 v^2 \sin^4(\theta/2)}. \tag{5.6}$$

This is the formula for relativistic electron scatted by Coulomb potential, and is called Mott formula.

Now we give an alternative derivation of the Mott formula, by considering the cross section of $e^-\mu^{-Z} \to e^-\mu^{-Z}$. When the mass of μ goes to infinity and the charge of μ is taken to be Ze, this cross section will reduces to Mott formula.

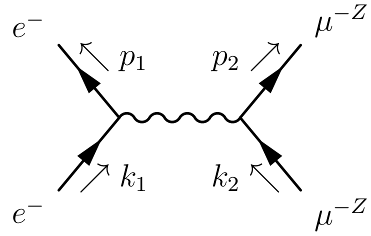


Figure 5.1: The scattering of an electron by a charged heavy particle μ^{-Z}. All initial momenta go inward and all final momenta go outward.

The corresponding Feynman diagram is shown in Figure 5.1, which reads,

$$\mathrm{i}\mathcal{M} = Z(-\mathrm{i}e)^2\bar{u}(p_1)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{t}\bar{U}(p_2)\gamma_\mu U(k_2), \tag{5.7}$$

where u is the spinor for electron and U is the spinor for muon, $t = (k_1 - p_1)^2$ is one of three Mandelstam variables. Then the squared amplitude with initial spin averaged and final spin summed is

$$\begin{aligned}\tfrac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2 &= \frac{Z^2e^4}{t^2}\,\mathrm{tr}\Big[\gamma^\mu(\not{k}_1+m)\gamma^\nu(\not{p}_1+m)\Big]\,\mathrm{tr}\Big[\gamma_\mu(\not{k}_2+M)\gamma_\nu(\not{p}_2+M)\Big]\\ &= \frac{Z^2e^4}{t^2}\Big[16m^2M^2 - 8M^2(k_1\cdot p_1) + 8(k_1\cdot p_2)(k_2\cdot p_1)\\ &\qquad - 8m^2(k_2\cdot p_2) + 8(k_1\cdot k_2)(p_1\cdot p_2)\Big].\end{aligned} \tag{5.8}$$

Note that the cross section is given by

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{2E_e2E_\mu|v_{\mathbf{k}_1}-v_{\mathbf{k}_2}|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\mathrm{CM}}}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big). \tag{5.9}$$

When the mass of μ goes to infinity, we have $E_\mu \simeq E_{\mathrm{CM}} \simeq M$, $v_{\mathbf{k}_2} \simeq 0$, and $|\mathbf{p}_1| \simeq |\mathbf{k}_1|$. Then the expression above can be simplified to

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\mathrm{CM}} = \frac{1}{16(2\pi)^2M^2}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big). \tag{5.10}$$

When $M \to \infty$, only terms proportional to M^2 are relevant in $|\mathcal{M}|^2$. To evaluate this squared amplitude further, we assign each momentum a specific value in CM frame,

$$\begin{aligned} k_1 &= (E,0,0,k), & p_1 &\simeq (E,\sin\theta,0,k\cos\theta),\\ k_2 &\simeq (M,0,0,-k), & p_2 &\simeq (M,-k\sin\theta,0,-k\cos\theta),\end{aligned} \tag{5.11}$$

then $t = (k_1 - p_1)^2 = 4k^2 \sin^2 \frac{\theta}{2}$, and,

$$\frac{1}{4} \sum |\mathrm{i}\mathcal{M}|^2 = \frac{Z^2 e^4 (1 - v^2 \sin^2 \frac{\theta}{2})}{k^2 v^2 \sin^2 \frac{\theta}{2}} M^2 + \mathcal{O}(M). \tag{5.12}$$

Substituting this into the cross section, and sending $M \to \infty$, we reach the Mott formula again,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{Z^2 \alpha^2 \left(1 - v^2 \sin^2 \frac{\theta}{2}\right)}{4|\mathbf{k}|^2 v^2 \sin^4(\theta/2)}. \tag{5.13}$$

5.2 Bhabha scattering

The Bhabha scattering is the process $e^+e^- \to e^+e^-$. At the tree level, it consists of two diagrams, as shown in Figure 5.2.

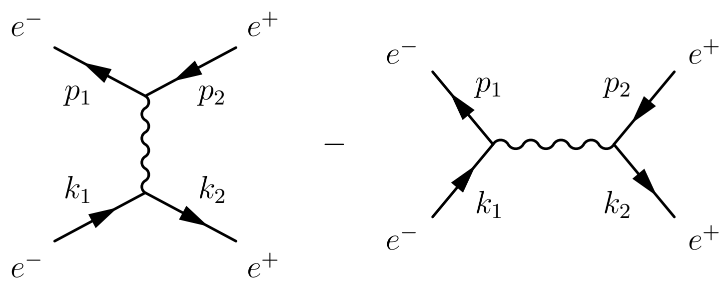


Figure 5.2: Bhabha scattering at tree level. All initial momenta go inward and all final momenta go outward.

The minus sign before the t-channel diagram comes from the exchange of two fermion field operators when contracting with in and out states. In fact, the s- and t-channel diagrams correspond to the following two ways of contraction, respectively,

$$\langle \mathbf{p}_1 \mathbf{p}_2 | \bar{\psi} \not{A} \psi \bar{\psi} \not{A} \psi | \mathbf{k}_1 \mathbf{k}_2 \rangle, \qquad \langle \mathbf{p}_1 \mathbf{p}_2 | \bar{\psi} \not{A} \psi \bar{\psi} \not{A} \psi | \mathbf{k}_1 \mathbf{k}_2 \rangle. \tag{5.14}$$

In the high energy limit, we can omit the mass of electrons, then the amplitude for the whole scattering process is,

$$i\mathcal{M} = (-\mathrm{i}e)^2 \left[\bar{v}(k_2) \gamma^\mu u(k_1) \frac{-\mathrm{i}}{s} \bar{u}(p_1) \gamma_\mu v(p_2) - \bar{u}(p_1) \gamma^\mu u(k_1) \frac{-\mathrm{i}}{t} \bar{v}(k_2) \gamma_\mu v(p_2) \right], \tag{5.15}$$

where we have used the Mandelstam variables s, t and u. They are defined as,

$$s = (k_1 + k_2)^2, \qquad t = (p_1 - k_1)^2, \qquad u = (p_2 - k_1)^2. \tag{5.16}$$

In the massless case, $k_1^2 = k_2^2 = p_1^2 = p_2^2 = 0$, thus we have,

$$s = 2k_1 \cdot k_2 = 2p_1 \cdot p_2, \qquad t = -2p_1 \cdot k_1 = -2p_2 \cdot k_2, \qquad u = -2p_2 \cdot k_1 = -2p_1 \cdot k_2. \tag{5.17}$$

We want to get the unpolarized cross section, thus we must average the ingoing spins and sum over outgoing spins. That is,

$$\frac{1}{4} \sum_{\text{spin}} |\mathcal{M}|^2 = \frac{e^4}{4s^2} \sum \left| \bar{v}(k_2) \gamma^\mu u(k_1) \bar{u}(p_1) \gamma_\mu v(p_2) \right|^2$$

$$
\begin{aligned}
&+\frac{e^4}{4t^2}\sum\Big|\bar{u}(p_1)\gamma^\mu u(k_1)\bar{v}(k_2)\gamma_\mu v(p_2)\Big|^2\\
&-\frac{e^4}{4st}\sum\Big[\bar{v}(p_2)\gamma_\mu u(p_1)\bar{u}(k_1)\gamma^\mu v(k_2)\bar{u}(p_1)\gamma^\nu u(k_1)\bar{v}(k_2)\gamma_\nu v(p_2)\ +\ \text{c.c.}\Big]\\
=&\frac{e^4}{4s^2}\operatorname{tr}(\not{k}_1\gamma^\mu\not{k}_2\gamma^\nu)\operatorname{tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu)+\frac{e^4}{4t^2}\operatorname{tr}(\not{k}_1\gamma^\mu\not{p}_1\gamma^\nu)\operatorname{tr}(\not{p}_2\gamma_\mu\not{k}_2\gamma_\nu)\\
&-\frac{e^4}{4st}\Big[\operatorname{tr}(\not{k}_1\gamma^\nu\not{k}_2\gamma_\mu\not{p}_2\gamma_\nu\not{p}_1\gamma^\mu)+\ \text{c.c.}\Big]\\
=&\frac{2e^4(u^2+t^2)}{s^2}+\frac{2e^4(u^2+s^2)}{t^2}+\frac{4e^4u^2}{st}\\
=&\,2e^4\bigg[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\bigg].
\end{aligned}
\tag{5.18}
$$

In the center-of-mass frame, we have $k_1^0=k_2^0\equiv k^0$, and $\mathbf{k}_1=-\mathbf{k}_2$, thus the total energy $E_{\text{CM}}^2=(k_1^0+k_2^0)^2=4k^2=s$. According to the formula for the cross section in the four identical particles' case (Eq.4.85):

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}}=\frac{1}{64\pi^2E_{\text{CM}}}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big),
\tag{5.19}
$$

thus

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}}=\frac{\alpha^2}{2s}\bigg[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\bigg],
\tag{5.20}
$$

where $\alpha=e^2/4\pi$ is the fine structure constant. We integrate this over the angle φ to get:

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\text{CM}}=\frac{\pi\alpha^2}{s}\bigg[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\bigg].
\tag{5.21}
$$

5.3 The spinor products (2)

In this problem we continue our study of spinor product method in last chapter. The formulae needed in the following are:

$$
u_L(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0},\qquad u_R(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}.
\tag{5.22}
$$

$$
s(p_1,p_2)=\bar{u}_R(p_1)u_L(p_2),\qquad t(p_1,p_2)=\bar{u}_L(p_1)u_R(p_2).
\tag{5.23}
$$

For detailed explanation for these relations, see Problem 3.3.

(a) Firstly, we prove the following relation,

$$
|s(p_1,p_2)|^2=2p_1\cdot p_2.
\tag{5.24}
$$

We make use of the another two relations,

$$
u_{L0}\bar{u}_{L0}=\frac{1-\gamma^5}{2}\not{k}_0,\qquad u_{R0}\bar{u}_{R0}\frac{1+\gamma^5}{2}\not{k}_0,
\tag{5.25}
$$

which are direct consequences of the familiar spin-sum formula $\sum u_0\bar{u}_0 = \not{k}_0$. We now generalize this to,

$$u_L(p)\bar{u}_L(p) = \frac{1-\gamma^5}{2}\not{p}, \qquad u_R(p)\bar{u}_R(p) = \frac{1+\gamma^5}{2}\not{p}. \tag{5.26}$$

We prove the first one:

$$\begin{aligned} u_L(p)\bar{u}_L(p) =& \frac{1}{2p\cdot k_0}\not{p}u_{R0}\bar{u}_{R0}\not{p} = \frac{1}{2p\cdot k_0}\not{p}\frac{1+\gamma^5}{2}\not{k}_0\not{p} \\ =& \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}\not{p}\not{k}_0\not{p} = \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}(2p\cdot k - \not{k}_0\not{p})\not{p} \\ =& \frac{1-\gamma^5}{2}\not{p} - \frac{1}{2p\cdot k_0}\frac{1-\gamma^5}{2}\not{k}_0 p^2 = \frac{1-\gamma^5}{2}\not{p}. \end{aligned} \tag{5.27}$$

The last equality holds because p is lightlike. Then we get,

$$\begin{aligned} |s(p_1,p_2)|^2 =& |\bar{u}_R(p_1)u_L(p_2)|^2 = \operatorname{tr}\Big(u_L(p_2)\bar{u}_L(p_2)u_R(p_1)\bar{u}_R(p_1)\Big) \\ =& \frac{1}{4}\operatorname{tr}\big((1-\gamma^5)\not{p}_2(1-\gamma^5)\not{p}_1\big) = 2p_1\cdot p_2. \end{aligned} \tag{5.28}$$

(b) Now we prove the relation,

$$\operatorname{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) = \operatorname{tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}), \tag{5.29}$$

where $\mu_i = 0,1,2,3,5$.

To make things easier, let us perform the proof in Weyl representation, without loss of generality. Then it's easy to check that

$$(\gamma^\mu)^T = \begin{cases} \gamma^\mu, & \mu = 0,2,5; \\ -\gamma^\mu, & \mu = 1,3. \end{cases} \tag{5.30}$$

Then, we define $M = \gamma^1\gamma^3$, and it can be easily shown that $M^{-1}\gamma^\mu M = (\gamma^\mu)^T$, and $M^{-1}M = 1$. Then we have,

$$\begin{aligned} \operatorname{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) =& \operatorname{tr}(M^{-1}\gamma^{\mu_1}MM^{-1}\gamma^{\mu_2}M\cdots M^{-1}\gamma^{\mu_n}M) \\ =& \operatorname{tr}\big[(\gamma^{\mu_1})^T(\gamma^{\mu_2})^T\cdots(\gamma^{\mu_n})^T\big] = \operatorname{tr}\big[(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1})^T\big] \\ =& \operatorname{tr}(\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}). \end{aligned} \tag{5.31}$$

With this formula in hand, we can derive the equality,

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2) = \bar{u}_R(p_2)\gamma^\mu u_R(p_1), \tag{5.32}$$

as follows,

$$\begin{aligned} \text{LHS} =& C\bar{u}_{R0}\not{p}_1\gamma^\mu\not{p}_2u_{R0} = C\operatorname{tr}(\not{p}_1\gamma^\mu\not{p}_2) \\ =& C\operatorname{tr}(\not{p}_2\gamma^\mu\not{p}_1) = C\bar{u}_{L0}\not{p}_2\gamma^\mu\not{p}_1u_{L0} = \text{RHS}, \end{aligned}$$

in which $C \equiv \big(2\sqrt{(p_1\cdot k_0)(p_2\cdot k_0)}\big)^{-1}$.

(c) The way of proving the Fierz identity

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma_\mu]_{ab} = 2\big[u_L(p_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_2)\big]_{ab} \tag{5.33}$$

has been indicated in P&S. The right hand side of this identity, as a Dirac matrix, which we denoted by M, can be written as a linear combination of 16 Γ matrices listed in Problem 3.6. In addition, it is easy to check directly that $\gamma^\mu M = -M\gamma^5$. Thus M must have the form

$$M = \Big(\frac{1-\gamma^5}{2}\Big)\gamma_\mu V^\mu + \Big(\frac{1+\gamma^5}{2}\Big)\gamma_\mu W^\mu.$$

Each of the coefficients V^μ and W^μ can be determined by projecting out the other one with the aid of trace technology, that is,

$$V^\mu = \frac{1}{2}\,\mathrm{tr}\Big[\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)M\Big] = \bar{u}_L(p_1)\gamma^\mu u_L(p_2), \tag{5.34}$$

$$W^\mu = \frac{1}{2}\,\mathrm{tr}\Big[\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)M\Big] = \bar{u}_R(p_2)\gamma^\mu u_R(p_1) = \bar{u}_L(p_1)\gamma^\mu u_L(p_2). \tag{5.35}$$

The last equality follows from (5.32). Substituting V^μ and W^μ back, we finally get the left hand side of the Fierz identity, which completes the proof.

(d) The amplitude for the process at leading order in α is given by,

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e^2)\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{s}\bar{v}_R(p_1)\gamma_\mu v_R(p_2). \tag{5.36}$$

To make use of the Fierz identity, we multiply (5.33), with the momenta variables changed to $p_1 \to k_1$ and $p_2 \to k_2$, by $\big[\bar{v}_R(p_1)\big]_a$ and $\big[v_R(p_2)\big]_b$, and also take account of (5.32), which leads to,

$$\begin{aligned}
&\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\bar{v}_R(p_1)\gamma_\mu v_R(p_2)\\
&= 2\big[\bar{v}_R(p_1)u_L(k_2)\bar{u}_L(k_1)v_R(p_2) + \bar{v}_R(p_1)u_R(k_1)\bar{u}_R(k_2)v_R(p_2)\big]\\
&= 2s(p_1,k_2)t(k_1,p_2).
\end{aligned} \tag{5.37}$$

Then,

$$|\mathrm{i}\mathcal{M}|^2 = \frac{4e^4}{s^2}|s(p_1,k_2)|^2|t(k_1,p_2)|^2 = \frac{16e^4}{s^2}(p_1\cdot k_2)(k_1\cdot p_2) = e^4(1+\cos\theta)^2, \tag{5.38}$$

and

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to \mu_L^+\mu_R^-) = \frac{|\mathrm{i}\mathcal{M}|^2}{64\pi^2 E_{\mathrm{cm}}} = \frac{\alpha^2}{4E_{\mathrm{cm}}}(1+\cos\theta)^2. \tag{5.39}$$

It is straightforward to work out the differential cross section for other polarized processes in similar ways. For instance,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to \mu_R^+\mu_L^-) = \frac{e^4|t(p_1,k_1)|^2|s(k_2,p_2)|^2}{64\pi^2 E_{\mathrm{cm}}} = \frac{\alpha^2}{4E_{\mathrm{cm}}}(1-\cos\theta)^2. \tag{5.40}$$

(e) Now we recalculate the Bhabha scattering studied in Problem 5.2, by evaluating all the polarized amplitudes. For instance,

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^+e_R^- \to e_L^+e_R^-) \\
&= (-\mathrm{i}e)^2\bigg[\bar{u}_R(k_2)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{s}\bar{v}_R(p_1)\gamma_\mu v_R(p_2) \\
&\qquad - \bar{u}_R(p_1)\gamma^\mu u_R(k_1)\frac{-\mathrm{i}}{t}\bar{v}_R(k_2)\gamma_\mu v_R(p_2)\bigg] \\
&= 2\mathrm{i}e^2\bigg[\frac{s(p_1,k_2)t(k_1,p_2)}{s} - \frac{s(k_2,p_1)t(k_1,p_2)}{t}\bigg]. \qquad (5.41)
\end{aligned}$$

Similarly,

$$\mathrm{i}\mathcal{M}(e_L^+e_R^- \to e_R^+e_L^-) = 2\mathrm{i}e^2\frac{t(p_1,k_1)s(k_2,p_2)}{s}, \qquad (5.42)$$

$$\mathrm{i}\mathcal{M}(e_R^+e_L^- \to e_L^+e_R^-) = 2\mathrm{i}e^2\frac{s(p_1,k_1)t(k_2,p_2)}{s}, \qquad (5.43)$$

$$\mathrm{i}\mathcal{M}(e_R^+e_L^- \to e_R^+e_L^-) = 2\mathrm{i}e^2\bigg[\frac{t(p_1,k_2)s(k_1,p_2)}{s} - \frac{t(k_2,p_1)s(k_1,p_2)}{t}\bigg], \qquad (5.44)$$

$$\mathrm{i}\mathcal{M}(e_R^+e_R^- \to e_R^+e_R^-) = 2\mathrm{i}e^2\frac{t(k_2,k_1)s(p_1,p_2)}{t}, \qquad (5.45)$$

$$\mathrm{i}\mathcal{M}(e_L^+e_L^- \to e_L^+e_L^-) = 2\mathrm{i}e^2\frac{s(k_2,k_1)t(p_1,p_2)}{t}. \qquad (5.46)$$

Squaring the amplitudes and including the kinematic factors, we find the polarized differential cross sections as,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to e_L^+e_R^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_L^- \to e_R^+e_L^-) = \frac{\alpha^2u^2}{2s}\Big(\frac{1}{s}+\frac{1}{t}\Big)^2, \qquad (5.47)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_R^- \to e_R^+e_L^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_L^- \to e_L^+e_R^-) = \frac{\alpha^2}{2s}\frac{t^2}{s^2}, \qquad (5.48)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_R^+e_R^- \to e_R^+e_R^-) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e_L^+e_L^- \to e_L^+e_L^-) = \frac{\alpha^2}{2s}\frac{s^2}{t^2}. \qquad (5.49)$$

Therefore we recover the result obtained in Problem 5.2,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(e^+e^- \to e^+e^-) = \frac{\alpha^2}{2s}\bigg[\frac{t^2}{s^2}+\frac{s^2}{t^2}+u^2\Big(\frac{1}{s}+\frac{1}{t}\Big)^2\bigg]. \qquad (5.50)$$

5.4 Positronium lifetimes

In this problem we study the decay of positronium (Ps) in its S and P states. To begin with, we recall the formalism developed for bound states with nonrelativistic quantum mechanics in P&S. The positronium state $|\mathrm{Ps}\rangle$, as a bound state of an electron-positron pair, can be represented in terms of electron and positron's state vectors, as,

$$|\mathrm{Ps}\rangle = \sqrt{2M_{\mathrm{P}}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi(\mathbf{k})C_{ab}\frac{1}{\sqrt{2m}}|e_a^-(\mathbf{k})\rangle\frac{1}{\sqrt{2m}}|e_b^+(-\mathbf{k})\rangle, \qquad (5.51)$$

where m is the electron's mass, $M_{\rm P}$ is the mass of the positronium, which can be taken to be $2m$ as a good approximation, a and b are spin labels, the coefficient C_{ab} depends on the spin configuration of $|{\rm Ps}\rangle$, and $\psi(\mathbf{k})$ is the momentum space wave function for the positronium in nonrelativistic quantum mechanics. In real space, we have,

$$\psi_{100}(r) = \sqrt{\frac{(\alpha m_{\rm r})^3}{\pi}} \exp(-\alpha m_{\rm r} r), \tag{5.52}$$

$$\psi_{21i}(r) = \sqrt{\frac{(\alpha m_{\rm r}/2)^5}{\pi}} x^i \exp(-\alpha m_{\rm r} r/2). \tag{5.53}$$

where $m_{\rm r} = m/2$ is the reduced mass. Then the amplitude of the decay process ${\rm Ps} \to 2\gamma$ can be represented in terms of the amplitude for the process $e^+e^- \to 2\gamma$ as,

$$\mathcal{M}({\rm Ps} \to 2\gamma) = \frac{1}{\sqrt{m}} \int \frac{{\rm d}^3 k}{(2\pi)^3} \psi(\mathbf{k}) C_{ab} \widehat{\mathcal{M}}\big(e_a^-(\mathbf{k}) e_b^+(-\mathbf{k}) \to 2\gamma\big). \tag{5.54}$$

We put a hat on the amplitude of $e^+e^- \to 2\gamma$. In the following we always use a hat to denote the amplitude of this process.

(a) In this part we study the decay of the S-state positronium. As stated above, we have to know the amplitude of the process $e^+e^- \to 2\gamma$, which is illustrated in Figure 5.3 with the B replaced with γ, and is given by,

$$\begin{aligned} {\rm i}\widehat{\mathcal{M}} &= (-{\rm i}e)^2 \epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2) \\ &\quad \times \bar{v}(k_2)\bigg[\gamma^\nu \frac{{\rm i}(\not{k}_1 - \not{p}_1 + m)}{(k_1-p_1)^2 - m^2}\gamma^\mu + \gamma^\mu \frac{{\rm i}(\not{k}_1 - \not{p}_2 + m)}{(k_1-p_2)^2 - m^2}\gamma^\nu\bigg] u(k_1), \end{aligned} \tag{5.55}$$

where the spinors can be written in terms of two-component spinors ξ and ξ' in the chiral representation as,

$$u(k_1) = \begin{pmatrix} \sqrt{k_1\cdot\sigma}\,\xi \\ \sqrt{k_1\cdot\bar\sigma}\,\xi \end{pmatrix}, \qquad v(k_2) = \begin{pmatrix} \sqrt{k_2\cdot\sigma}\,\xi' \\ -\sqrt{k_2\cdot\bar\sigma}\,\xi' \end{pmatrix}. \tag{5.56}$$

We also write γ^μ as,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (1, \sigma^i)$ and $\bar\sigma^\mu = (1, -\sigma^i)$ with σ^i the three Pauli matrices. Then the amplitude can be brought into the following form,

$${\rm i}\widehat{\mathcal{M}} = -{\rm i}e^2 \epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\xi'^\dagger \bigg[\frac{\Gamma_t^{\mu\nu}}{(k_1-p_1)^2 - m^2} + \frac{\Gamma_u^{\mu\nu}}{(k_1-p_2)^2-m^2}\bigg]\xi, \tag{5.57}$$

with

$$\begin{aligned} \Gamma_t^{\mu\nu} = &\Big(\sqrt{k_2\cdot\sigma}\,\bar\sigma^\nu\sigma^\mu\sqrt{k_1\cdot\bar\sigma} - \sqrt{k_2\cdot\bar\sigma}\,\sigma^\nu\bar\sigma^\mu\sqrt{k_1\cdot\sigma}\Big)m \\ &+ \Big(\sqrt{k_2\cdot\sigma}\,\bar\sigma^\nu\sigma^\lambda\bar\sigma^\mu\sqrt{k_1\cdot\sigma} - \sqrt{k_2\cdot\bar\sigma}\,\sigma^\nu\bar\sigma^\lambda\sigma^\mu\sqrt{k_1\cdot\bar\sigma}\Big)(k_1-p_1)_\lambda, \end{aligned}$$

$$\Gamma_u^{\mu\nu} = \Big(\sqrt{k_2\cdot\sigma}\bar{\sigma}^\mu\sigma^\nu\sqrt{k_1\cdot\bar{\sigma}} - \sqrt{k_2\cdot\bar{\sigma}}\sigma^\mu\bar{\sigma}^\nu\sqrt{k_1\cdot\sigma}\Big)m \\ + \Big(\sqrt{k_2\cdot\sigma}\bar{\sigma}^\mu\sigma^\lambda\bar{\sigma}^\nu\sqrt{k_1\cdot\sigma} - \sqrt{k_2\cdot\bar{\sigma}}\sigma^\mu\bar{\sigma}^\lambda\sigma^\nu\sqrt{k_1\cdot\bar{\sigma}}\Big)(k_1-p_2)_\lambda.$$

In the rest of the part (a), we take the nonrelativistic limit, with the momenta chosen to be

$$k_1^\mu = k_2^\mu = (m,0,0,0), \qquad p_1^\mu = (m,0,0,m), \qquad p_2^\mu = (m,0,0,-m). \tag{5.58}$$

Accordingly, we can assign the polarization vectors for final photons to be,

$$\epsilon_\pm^\mu(p_1) = \tfrac{1}{\sqrt{2}}(0,1,\pm\mathrm{i},0), \qquad \epsilon_\pm^\mu(p_2) = \tfrac{1}{\sqrt{2}}(0,-1,\pm\mathrm{i},0). \tag{5.59}$$

Now substituting the momenta (5.58) into (5.57), noticing that $\sqrt{k_i\cdot\sigma} = \sqrt{k_i\cdot\bar{\sigma}} = \sqrt{m}$ ($i = 1,2$), and $(k_1-p_1)^2 = (k_1-p_2)^2 = -m^2$, and also using the trick that one can freely make the substitution $\bar{\sigma}^\mu \to -\sigma^\mu$ since the temporal component of the polarization vectors ϵ_μ always vanishes, we get a much more simplified expression,

$$\mathrm{i}\widehat{\mathcal{M}} = \mathrm{i}e^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\xi'^\dagger\big(\sigma^\nu\sigma^3\sigma^\mu - \sigma^\mu\sigma^3\sigma^\nu\big)\xi. \tag{5.60}$$

The positronium can lie in spin-0 (singlet) state or spin-1 (triplit) state. In the former case, we specify the polarizations of final photons in all possible ways, and also make the substitution $\xi\xi'^\dagger \to \frac{1}{\sqrt{2}}$ (See (5.49) of P&S), which leads to,

$$\mathrm{i}\widehat{\mathcal{M}}_{++}^s = -\mathrm{i}\widehat{\mathcal{M}}_{--}^s = \mathrm{i}2\sqrt{2}e^2, \qquad \mathrm{i}\widehat{\mathcal{M}}_{+-}^s = \mathrm{i}\widehat{\mathcal{M}}_{-+}^s = 0, \tag{5.61}$$

where the subscripts denote final photons' polarizations and s means singlet. We show the mid-step for calculating $\mathrm{i}\mathcal{M}_{++}^s$ as an example,

$$\mathrm{i}\widehat{\mathcal{M}}_{++}^s = \frac{\mathrm{i}e^2}{2}\,\mathrm{tr}\Big[\Big((\sigma^1+\mathrm{i}\sigma^2)\sigma^3(-\sigma^1+\mathrm{i}\sigma^2) - (-\sigma^1+\mathrm{i}\sigma^2)\sigma^3(\sigma^1+\mathrm{i}\sigma^2)\Big)\xi\xi'^\dagger\Big] = \mathrm{i}2\sqrt{2}e^2.$$

In the same way, we can calculate the case of triplet initial state. This time, we make the substitution $\xi\xi'^\dagger \to \mathbf{n}\cdot\boldsymbol{\sigma}/\sqrt{2}$, with $\mathbf{n} = (\hat{x}\pm\mathrm{i}\hat{y})/\sqrt{2}$ or $\mathbf{n} = \hat{z}$, corresponding to three independent polarizations. But it is straightforward to show that the amplitudes with these initial polarizations all vanish, which is consistent with our earlier results by using symmetry arguments in Problem 3.8.

Therefore it is enough to consider the singlet state only. The amplitude for the decay of a positronium in its ${}^1S_0$ state into 2γ then follows directly from (5.54), as

$$\mathcal{M}_{\pm\pm}({}^1S_0\to 2\gamma) = \frac{\psi(\mathbf{x}=0)}{\sqrt{m}}\widehat{\mathcal{M}}_{\pm\pm}^s, \tag{5.62}$$

where $\psi(\mathbf{x}=0) = \sqrt{(m\alpha/2)^3/\pi}$ according to (5.52). Then the squared amplitude with final photons' polarizations summed is

$$\sum_{\text{spin}}\big|\mathcal{M}({}^1S_0\to 2\gamma)\big|^2 = \frac{|\psi(0)|^2}{2m}\Big(|\mathcal{M}_{++}^s|^2 + |\mathcal{M}_{--}^s|^2\Big) = 16\pi\alpha^5m^2. \tag{5.63}$$

Finally we find the decay width of the process $\text{Ps}(^1S_0) \to 2\gamma$, to be

$$\begin{aligned}\Gamma(^1S_0 \to 2\gamma) &= \frac{1}{2}\frac{1}{4m}\int \frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_1 2E_2}\sum\big|\mathcal{M}(^1S_0 \to 2\gamma)\big|^2(2\pi)^4\delta^{(4)}(p_{\text{Ps}} - p_1 - p_2)\\ &= \frac{1}{2}\frac{1}{4m}\int \frac{\mathrm{d}^3p_1}{(2\pi)^3 4m^2}\sum\big|\mathcal{M}(^1S_0 \to 2\gamma)\big|^2(2\pi)\delta(m - E_1)\\ &= \frac{1}{2}\alpha^5 m, \end{aligned} \tag{5.64}$$

where an additional factor of 1/2 follows from the fact that the two photons in the final state are identical particles.

(b) To study the decay of P state ($l = 1$) positronium, we should keep one power of 3-momenta of initial electron and positron. Thus we set the momenta of initial and final particles, and also the polarization vectors of the latter, in $e^-e^+ \to 2\gamma$, to be

$$\begin{aligned} &k_1^\mu = (E, 0, 0, k), && k_2^\mu = (E, 0, 0, -k),\\ &p_1^\mu = (E, E\sin\theta, 0, E\cos\theta), && p_2^\mu = (E, -E\sin\theta, 0, -E\cos\theta),\\ &\epsilon_\pm^\mu(p_1) = \tfrac{1}{\sqrt{2}}(0, \cos\theta, \pm\mathrm{i}, -\sin\theta), && \epsilon_\pm^\mu(p_2) = \tfrac{1}{\sqrt{2}}(0, -\cos\theta, \pm\mathrm{i}, \sin\theta). \end{aligned} \tag{5.65}$$

Here we have the approximate expression up to linear order in k:

$$\begin{aligned} \sqrt{k_1\cdot\sigma} = \sqrt{k_2\cdot\bar\sigma} &= \sqrt{m} - \frac{k}{2\sqrt{m}}\sigma^3 + \mathcal{O}(k^2),\\ \sqrt{k_2\cdot\sigma} = \sqrt{k_1\cdot\bar\sigma} &= \sqrt{m} + \frac{k}{2\sqrt{m}}\sigma^3 + \mathcal{O}(k^2),\\ \frac{1}{(k_1 - p_1)^2 - m^2} &= -\frac{1}{2m^2} - \frac{k\cos\theta}{2m^3} + \mathcal{O}(k^2),\\ \frac{1}{(k_1 - p_2)^2 - m^2} &= -\frac{1}{2m^2} + \frac{k\cos\theta}{2m^3} + \mathcal{O}(k^2). \end{aligned}$$

Consequently,

$$\begin{aligned} \Gamma_t^{\mu\nu} &= 2m^2\sigma^\nu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\mu - mk\big(\sigma^3\sigma^\nu\sigma^\mu + \sigma^\nu\sigma^\mu\sigma^3 + 2\sigma^\nu\sigma^3\sigma^\mu\big) + \mathcal{O}(k^2),\\ \Gamma_u^{\mu\nu} &= -2m^2\sigma^\mu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\nu - mk\big(\sigma^3\sigma^\mu\sigma^\nu + \sigma^\mu\sigma^\nu\sigma^3 + 2\sigma^\mu\sigma^3\sigma^\nu\big) + \mathcal{O}(k^2), \end{aligned}$$

where we use the shorthand notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. We can use these expansion to find the terms in the amplitude $\mathrm{i}\widehat{\mathcal{M}}$ of linear order in k, to be

$$\begin{aligned} \mathrm{i}\widehat{\mathcal{M}}\Big|_{\mathcal{O}(k)} =& -\mathrm{i}e^2\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2)\frac{k}{2m}\xi'^\dagger\Big[-2c_\theta\sigma^\mu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\nu - 2c_\theta\sigma^\nu(\sigma^1 s_\theta + \sigma^3 c_\theta)\sigma^\mu\\ &+ \big(\sigma^3\sigma^\mu\sigma^\nu + \sigma^\mu\sigma^\nu\sigma^3 + 2\sigma^\mu\sigma^3\sigma^\nu\big) + \big(\sigma^3\sigma^\nu\sigma^\mu + \sigma^\nu\sigma^\mu\sigma^3 + 2\sigma^\nu\sigma^3\sigma^\mu\big)\Big]\xi, \end{aligned} \tag{5.66}$$

Feeding in the polarization vectors of photons, and also make the substitution $\xi\xi'^\dagger \to \mathbf{n}\cdot\boldsymbol{\sigma}/\sqrt{2}$ or $1/\sqrt{2}$ for triplet and singlet positronium, respectively, as done in last part, we get

$$\mathrm{i}\widehat{\mathcal{M}}_{\pm\pm}^{\downarrow\downarrow}\big|_{\mathcal{O}(k)} = 0, \qquad \mathrm{i}\widehat{\mathcal{M}}_{\pm\mp}^{\downarrow\downarrow}\big|_{\mathcal{O}(k)} = -\mathrm{i}2s_\theta(\mp 1 + c_\theta)e^2k/m,$$

$$\begin{aligned}
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow+\downarrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = \mathrm{i}2\sqrt{2}e^2k/m, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow+\downarrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = \mathrm{i}2\sqrt{2}s_\theta^2e^2k/m,\\
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = 0, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = -\mathrm{i}2s_\theta(\pm1+c_\theta)e^2k/m,\\
&\mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow-\downarrow\uparrow}_{\pm\pm}|_{\mathcal{O}(k)} = 0, && \mathrm{i}\widehat{\mathcal{M}}^{\uparrow\downarrow-\downarrow\uparrow}_{\pm\mp}|_{\mathcal{O}(k)} = 0.
\end{aligned} \tag{5.67}$$

The vanishing results in the last line indicate that $S = 0$ state of P-wave positronium cannot decay to two photons.

(c) Now we prove that the state,

$$|B(\mathbf{k})\rangle = \sqrt{2M_\mathrm{P}}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\psi_i(\mathbf{p})a^\dagger_{\mathbf{p}+\mathbf{k}/2}\Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2}|0\rangle \tag{5.68}$$

, is a properly normalized state for the P-wave positronium. In fact,

$$\begin{aligned}
\langle B(\mathbf{k})|B(\mathbf{k})\rangle &= 2M_\mathrm{P}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi_j^*(\mathbf{p}')\psi_i(\mathbf{p})\\
&\qquad\times\langle 0|b_{-\mathbf{p}'+\mathbf{k}/2}\Sigma^{j\dagger}a_{\mathbf{p}'+\mathbf{k}/2}a^\dagger_{\mathbf{p}+\mathbf{k}/2}\Sigma^i b^\dagger_{-\mathbf{p}'+\mathbf{k}/2}|0\rangle\\
&= 2M_\mathrm{P}\int\frac{\mathrm{d}^3p'}{(2\pi)^3}\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi_j^*(\mathbf{p}')\psi_i(\mathbf{p})\\
&\qquad\times\langle 0|b_{-\mathbf{p}'+\mathbf{k}/2}\Sigma^{j\dagger}\Sigma^i b^\dagger_{-\mathbf{p}'+\mathbf{k}/2}|0\rangle(2\pi)^3\delta^{(3)}(\mathbf{p}'-\mathbf{p})\\
&= 2M_\mathrm{P}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi_j^*(\mathbf{p})\psi_i(\mathbf{p})\langle 0|b_{-\mathbf{p}+\mathbf{k}/2}\Sigma^{j\dagger}\Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2}|0\rangle\\
&= 2M_\mathrm{P}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\psi_j^*(\mathbf{p})\psi_i(\mathbf{p})\langle 0|\,\mathrm{tr}\,(\Sigma^{j\dagger}\Sigma^i)|0\rangle(2\pi)^3\delta^{(3)}(\mathbf{0})\\
&= 2M_\mathrm{P}\cdot(2\pi)^3\delta^{(3)}(\mathbf{0}),
\end{aligned} \tag{5.69}$$

which is precisely the needed normalization of a state. In this calculation we have used the anticommutation relations of creation and annihilation operators, as well as the normalization of the wave function and the Σ matrices.

(d) Now we evaluate the partial decay rate of the $S = 1$ P-wave positronium of definite J into two photons. The states for the positronium is presented in (c), with the Σ matrices chosen as

$$\Sigma = \begin{cases} \dfrac{1}{\sqrt{6}}\sigma^i, & J = 0,\\ \dfrac{1}{2}\epsilon^{ijk}n^j\sigma^k, & J = 1,\\ \dfrac{1}{\sqrt{3}}h^{ij}\sigma^j, & J = 2,\end{cases} \tag{5.70}$$

and the wave function given by (5.53).

Firstly, consider the $J = 0$ state, in which case we have,

$$\mathrm{i}\mathcal{M}({}^3P_0\to\gamma_\alpha\gamma_\beta) = \frac{1}{\sqrt{m}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi_i(\mathbf{k})\Big(\frac{1}{\sqrt{6}}\sigma^i\Big)_{ab}\mathrm{i}\widehat{\mathcal{M}}\big(e^-_a(\mathbf{k})e^+_b(-\mathbf{k})\to\gamma_\alpha\gamma_\beta\big), \tag{5.71}$$

where $\alpha, \beta = +$ or $-$ are labels of photons' polarizations and $a, b = \uparrow$ or $\downarrow$ are spinor indices. For amplitude $\mathrm{i}\widehat{\mathcal{M}}$, we only need the terms linear in $\mathbf{k}$, as listed in (5.67). Let us rewrite this as,

$$\mathrm{i}\widehat{\mathcal{M}}\big(e_a^-(\mathbf{k})e_b^+(-\mathbf{k}) \to \gamma_\alpha\gamma_\beta\big) = F_{\alpha\beta,i}^{ab} k^i.$$

In the same way, the wave function can also be put into the form of $\psi_i(\mathbf{x}) = x^i f(r)$, with $r = |\mathbf{x}|$. Then the integration above can be carried out to be,

$$\begin{aligned}\mathrm{i}\mathcal{M}(^3P_0 \to \gamma_\alpha\gamma_\beta) &= \frac{\mathrm{i}}{\sqrt{6m}}\sigma_{ab}^i F_{\alpha\beta,j}^{ab}\left[\mathrm{i}\frac{\partial}{\partial x^j}\psi^i(\mathbf{x})\right]_{\mathbf{x}=0} \\ &= \frac{\mathrm{i}}{\sqrt{6m}}\sigma_{ab}^i F_{\alpha\beta,i}^{ab} f(0). \end{aligned} \tag{5.72}$$

On the other hand, we have chose the direction of $\mathbf{k}$ to be in the x^3-axis, then $F_{\alpha\beta,1}^{ab} = F_{\alpha\beta,2}^{ab} = 0$ as a consequence. Therefore,

$$\mathrm{i}\mathcal{M}(^3P_0 \to \gamma_\pm\gamma_\mp) = \frac{\mathrm{i}}{\sqrt{6m}} f(0)\big(F_{\pm\mp,3}^{\uparrow\uparrow} - F_{\pm\mp,3}^{\downarrow\downarrow}\big) = \pm\sqrt{\frac{\pi\alpha^7}{24}}\, m\sin\theta. \tag{5.73}$$

Square these amplitudes, sum over the photons' polarizations, and finish the phase space integration in the same way as what we did in (a), we finally get the partial decay rate of the $J = 0$ P-wave positronium into two photons to be,

$$\Gamma(^3P_0) = \frac{1}{576}\alpha^7 m. \tag{5.74}$$

The positronium in $^3P_1$ state, namely the case $J = 1$, cannot decay into two photons by the conservation of the angular momentum, since the total angular momentum of two physical photons cannot be 1. Therefore let us turn to the case of $J = 2$. In this case we should average over the initial polarizations of the positronium, which can be represented by the symmetric and traceless polarization tensors h_n^{ij}, with $n = 1, 2, \cdots, 5$ the labeled of 5 independent polarizations. Let us choose these tensors to be,

$$\begin{aligned} h_1^{ij} &= \tfrac{1}{\sqrt{2}}(\delta^{i2}\delta^{j3} + \delta^{i3}\delta^{j2}), & h_2^{ij} &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j3} + \delta^{i3}\delta^{j1}), \\ h_3^{ij} &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j2} + \delta^{i2}\delta^{j1}), & h_4^{ij} &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j1} - \delta^{i2}\delta^{j2}), \\ h_5^{ij} &= \tfrac{1}{\sqrt{2}}(\delta^{i1}\delta^{j1} - \delta^{i3}\delta^{j3}). \end{aligned} \tag{5.75}$$

Then the decay amplitude for a specific polarization of $J = 2$ Ps can be represented as,

$$\begin{aligned}\mathrm{i}\mathcal{M}_n(^3P_2 \to \gamma_\alpha\gamma_\beta) &= \frac{1}{\sqrt{m}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\psi_i(\mathbf{k})\Big(\frac{1}{\sqrt{3}}h_n^{ij}\sigma^j\Big)_{ab}\mathrm{i}\widehat{\mathcal{M}}\big(e_a^-(\mathbf{k})e_b^+(-\mathbf{k}) \to \gamma_\alpha\gamma_\beta\big) \\ &= \frac{1}{\sqrt{3m}}h_n^{ij}\sigma_{ab}^j F_{\alpha\beta,i}^{ab} f(0). \end{aligned} \tag{5.76}$$

Now substituting all stuffs in, we find the nonvanishing components of the decay amplitude to be,

$$\mathrm{i}\mathcal{M}_2(^3P_2 \to \gamma_\pm\gamma_\pm) = \sqrt{\frac{\pi\alpha^7}{48}}\, im,$$

$$\mathrm{i}\mathcal{M}_2(^3P_2 \to \gamma_\pm\gamma_\mp) = \sqrt{\frac{\pi\alpha^7}{48}} im \sin^2\theta,$$
$$\mathrm{i}\mathcal{M}_5(^3P_2 \to \gamma_\pm\gamma_\mp) = \mp 2\sqrt{\frac{\pi\alpha^7}{48}} im \sin^2\theta. \tag{5.77}$$

Squaring these amplitudes, summing over photon's polarizations and averaging the initial polarization of the positronium (by dividing the squared and summed amplitude by 5), we get,

$$\frac{1}{5}\sum_{\text{spin}} \left|\mathcal{M}_n(^3P_2 \to 2\gamma)\right|^2 = \frac{\pi\alpha^7 m^2}{120}(1+\sin^2\theta + 4\sin^4\theta). \tag{5.78}$$

Finally, we finish the phase space integration and get the partial decay rate of $^3P_2$ positronium into 2 photons to be

$$\Gamma(^3P_2) = \frac{19}{19200}\alpha^7 m. \tag{5.79}$$

5.5 Physics of a massive vector boson

In this problem, the mass of electron is always set to zero.

(a) We firstly compute the cross section $\sigma(e^+e^- \to B)$ and the decay rate $\Gamma(B \to e^+e^-)$. For the cross section, the squared amplitude can be easily found to be

$$\frac{1}{4}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spin}} \left|\mathrm{i}g\epsilon_\mu^{*(i)}\bar{v}(p')\gamma^\mu u(p)\right|^2 = 2g^2(p\cdot p'). \tag{5.80}$$

Note that we have set the mass of electrons to be zero. Then the cross section can be deduced from (4.79). Let's take the initial momenta to be,

$$p = \tfrac{1}{2}(E,0,0,E), \qquad p' = \tfrac{1}{2}(E,0,0,-E), \tag{5.81}$$

with E being the center-of-mass energy. Then it's easy to get,

$$\sigma = \frac{g^2}{4E}(2\pi)\delta(M_B - E) = \frac{g^2}{4E}(2\pi)2M_B\delta(M_B^2 - s) = \pi g^2\delta(M_B^2 - s), \tag{5.82}$$

where $s = E^2$.

To deduce the decay rate, we should average polarizations of massive vector B instead of two electrons. Thus the squared amplitude in this case reads,

$$\frac{1}{3}\sum_{\text{spin}} |\mathrm{i}\mathcal{M}|^2 = \frac{8}{3}g^2(p\cdot p'). \tag{5.83}$$

The decay rate can be found from (4.86),

$$\Gamma = \frac{1}{2M_B}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3p'}{2\pi^3}\frac{1}{2E_{\mathbf{p}}}\frac{1}{2E_{\mathbf{p}'}}\Big(\tfrac{1}{3}\sum|\mathcal{M}|^2\Big)(2\pi)^4\delta^{(4)}(p_B - p - p')$$

$$
\begin{aligned}
&=\frac{1}{2M_B}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{4E_{\mathbf{p}}^2}\Big(\tfrac{16}{3}g^2E_{\mathbf{p}}^2\Big)(2\pi)\delta(M_B-2E_{\mathbf{p}})\\
&=\frac{4\pi}{(2\pi)^2 2M_B}\int\mathrm{d}p\,\tfrac{4}{3}g^2E_{\mathbf{p}}^2\tfrac{1}{2}\delta(\tfrac{1}{2}M_B-E_{\mathbf{p}})=\frac{g^2M_B}{12\pi}.
\end{aligned}\tag{5.84}
$$

We see the cross section and the decay rate satisfy the following relation, as expected,

$$
\sigma(e^+e^-\to B)=\frac{12\pi^2}{M_B}\Gamma(B\to e^+e^-)\delta(s-M^2).\tag{5.85}
$$

(b) Now we calculate the cross section $\sigma(e^-e^+\to\gamma+B)$ in COM frame. The related diagrams are shown in Figure 5.3.

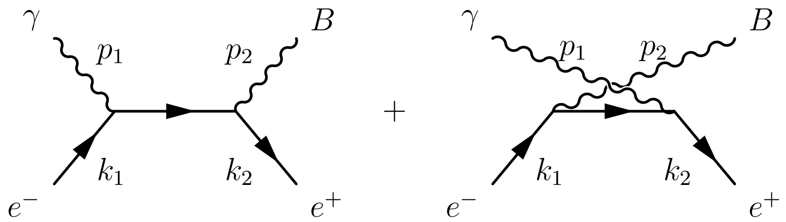


Figure 5.3: The tree diagrams of the process $e^-e^+\to\gamma+B$. All initial momenta go inward and all final momenta go outward.

The amplitude reads,

$$
\mathrm{i}\mathcal{M}=(-\mathrm{i}e)(-\mathrm{i}g)\epsilon_\mu^*(p_1)e_\nu^*(p_2)\bar v(k_2)\Big[\gamma^\nu\frac{\mathrm{i}}{\not k_1-\not p_1}\gamma^\mu+\gamma^\mu\frac{\mathrm{i}}{\not k_1-\not p_2}\gamma^\nu\Big]u(k_1),\tag{5.86}
$$

where ϵ_μ is the polarization of photon while e_μ is the polarization for B. Now we square this amplitude,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2=&\ \frac{1}{4}e^2g^2g_{\mu\rho}g_{\nu\sigma}\operatorname{tr}\bigg[\Big(\frac{\gamma^\nu(\not k_1-\not p_1)\gamma^\mu}{t}+\frac{\gamma^\mu(\not p_1-\not k_2)\gamma^\nu}{u}\Big)\not k_1\\
&\times\Big(\frac{\gamma^\rho(\not k_1-\not p_1)\gamma^\sigma}{t}+\frac{\gamma^\sigma(\not p_1-\not k_2)\gamma^\rho}{u}\Big)\not k_2\bigg]\\
=&\ 8e^2g^2\bigg[\frac{(k_1\cdot p_1)(k_2\cdot p_1)}{t^2}+\frac{(k_1\cdot p_1)(k_2\cdot p_1)}{u^2}\\
&+\frac{2(k_1\cdot k_2)(k_1\cdot k_2-k_1\cdot p_1-k_2\cdot p_1)}{tu}\bigg]\\
=&\ 2e^2g^2\bigg[\frac{u}{t}+\frac{t}{u}+\frac{2s(s+t+u)}{tu}\bigg]\\
=&\ 2e^2g^2\bigg[\frac{u}{t}+\frac{t}{u}+\frac{2sM_B^2}{tu}\bigg].
\end{aligned}\tag{5.87}
$$

Then the cross section can be evaluated as,

$$
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}}=\frac{1}{2E_{\mathbf{k}_1}2E_{\mathbf{k}_2}|v_{\mathbf{k}_1}-v_{\mathbf{k}_2}|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\text{CM}}}\Big(\tfrac{1}{4}\textstyle\sum|\mathcal{M}|^2\Big)
$$

$$=\frac{e^2g^2}{32\pi^2 s}\Big(1-\frac{M_B^2}{s}\Big)\Big[\frac{u}{t}+\frac{t}{u}+\frac{2sM_B^2}{tu}\Big]. \tag{5.88}$$

We can also write this differential cross section in terms of squared COM energy s and scattering angle θ. To do this, we note,

$$s=E_{\text{CM}}^2,\qquad t=(M_B^2-E_{\text{CM}}^2)\sin^2\tfrac{\theta}{2},\qquad u=(M_B^2-E_{\text{CM}}^2)\cos^2\tfrac{\theta}{2}. \tag{5.89}$$

Then we have,

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}}=\frac{e^2g^2(1-M_B^2/s)}{16\pi^2 s\sin^2\theta}\Big[1+\cos^2\theta+\frac{4sM_B^2}{(s-M_B^2)^2}\Big], \tag{5.90}$$

and,

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\text{CM}}=\frac{\alpha g^2(1-M_B^2/s)}{2s\sin^2\theta}\Big[1+\cos^2\theta+\frac{4sM_B^2}{(s-M_B^2)^2}\Big]. \tag{5.91}$$

(c) The differential cross obtained in (b) diverges when $\theta\to 0$ or $\theta\to\pi$. Now let us study the former case, namely $\theta\to 0$.

If we cut of the integral from $\theta_c^2\simeq m_e^2/s$, then we have,

$$\begin{aligned}\int_{\theta_c}\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\text{CM}}\sin\theta\mathrm{d}\theta &\simeq \frac{\alpha g^2(1-M_B^2/s)}{2s}\Big[2+\frac{4sM_B^2}{(s-M_B^2)^2}\Big]\int^{1-m_e^2/s}\frac{\mathrm{d}t}{1-t^2}\\ &\simeq \frac{\alpha g^2(1-M_B^2/s)}{4s}\Big[2+\frac{4sM_B^2}{(s-M_B^2)^2}\Big]\log\Big(\frac{s}{m_e^2}\Big)\\ &=\frac{\alpha g^2}{2}\frac{1+M_B^4/s^2}{s-M_B^2}\log\Big(\frac{s}{m_e^2}\Big).\end{aligned} \tag{5.92}$$

Now we calculate the following expression,

$$\begin{aligned}&\int_0^1 \mathrm{d}x\, f(x)\sigma(e^+e^-\to B)\big|_{E_{\text{CM}}=(1-x)s}\\ =&\int_0^1\mathrm{d}x\Big[\frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x}\log\Big(\frac{s}{m_e^2}\Big)\Big]\pi g^2\delta\big(M_B^2-(1-x)s\big)\\ =&\ \frac{\alpha g^2}{2}\frac{1+M_B^4/s^2}{s-M_B^2}\log\Big(\frac{s}{m_e^2}\Big).\end{aligned} \tag{5.93}$$

5.6 The spinor products (3)

This problem generalize the spinor product formalism to the processes involving external photons.

(a) Firstly we can represent photon's polarization vectors in terms of spinors of definite helicity. Let the momentum of the photon be k, and p be a lightlike momentum such that $p\cdot k\neq 0$. Then, the polarization vector $\epsilon_\pm^\mu(k)$ of the photon can be taken to be,

$$\epsilon_+^\mu(k)=\frac{1}{\sqrt{4p\cdot k}}\bar{u}_R(k)\gamma^\mu u_R(p),\qquad \epsilon_-^\mu(k)=\frac{1}{\sqrt{4p\cdot k}}\bar{u}_L(k)\gamma^\mu u_L(p), \tag{5.94}$$

where the spinors $u_{L,R}(k)$ have been introduced in Problems 3.3 and 5.3. Now we use this choice to calculate the polarization sum,

$$\begin{aligned}
&\epsilon_+^\mu(k)\epsilon_+^{\nu*}(k)+\epsilon_-^\mu(k)\epsilon_-^{\nu*}(k)\\
=&\frac{1}{4p\cdot k}\Big[\bar u_R(k)\gamma^\mu u_R(p)\bar u_R(p)\gamma^\nu u_R(k)+\bar u_L(k)\gamma^\mu u_L(p)\bar u_L(p)\gamma^\nu u_L(k)\Big]\\
=&\frac{1}{4p\cdot k}\,\mathrm{tr}\,\big[\not p\gamma^\nu\not k\gamma^\mu\big]=-g^{\mu\nu}+\frac{p^\mu k^\nu+p^\nu k^\mu}{p\cdot k}.
\end{aligned}\tag{5.95}$$

When dotted into an amplitude with external photon, the second term of the result vanishes. This justifies the definitions above for photon's polarization vectors.

(b) Now we apply the formalism to the process $e^+e^-\to 2\gamma$ in the massless limit. The relevant diagrams are similar to those in Figure 5.3, except that one should replace the label 'B' by 'γ'. To simplify expressions, we introduce the standard shorthand notations as follows:

$$p\rangle=u_R(p),\qquad p]=u_L(p),\qquad \langle p=\bar u_L(p),\qquad [p=\bar u_R(p).\tag{5.96}$$

Then the spin products become $s(p_1,p_2)=[p_1p_2]$ and $t(p_1,p_2)=\langle p_1p_2\rangle$. Various expressions get simplified with this notation. For example, the Fierz identity (5.37) now reads $[k_2\gamma^\mu k_1\rangle[p_1\gamma_\mu p_2\rangle=2[p_1k_2]\langle k_1p_2\rangle$. Similarly, we also have $\langle k_1\gamma^\mu k_2]\langle p_1\gamma_\mu p_2]=2\langle k_1p_1\rangle[p_2k_2]$.

Now we write down the expression for tree amplitude of $e_R^+e_L^-\to\gamma_R\gamma_L$. For illustration, we still keep the original expression as well as all explicit mid-steps. The auxiliary lightlike momenta used in the polarization vectors are arbitrarily chosen such that the calculation can be mostly simplified.

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_R^+e_L^-\to\gamma_L\gamma_R)\\
=&(-\mathrm{i}e)^2\epsilon^*_{-\mu}(p_1)\epsilon^*_{+\nu}(p_2)\bar u_L(k_2)\left[\gamma^\nu\frac{\mathrm{i}}{\not k_1-\not p_1}\gamma^\mu+\gamma^\mu\frac{\mathrm{i}}{\not k_1-\not p_1}\gamma^\nu\right]u_L(k_1)\\
=&-\mathrm{i}e^2\frac{\langle k_2\gamma_\mu p_1][k_1\gamma_\nu p_2\rangle}{4\sqrt{(k_2\cdot p_1)(k_1\cdot p_2)}}\left[\frac{\langle k_2\gamma^\nu(\not k_1-\not p_1)\gamma^\mu k_1]}{t}+\frac{\langle k_2\gamma^\mu(\not k_1-\not p_2)\gamma^\nu k_1]}{u}\right]\\
=&-\mathrm{i}e^2\frac{\langle k_2\gamma_\mu p_1][k_1\gamma_\nu p_2\rangle}{2u}\left[\frac{\langle k_2\gamma^\nu k_1]\langle k_1\gamma^\mu k_1]-\langle k_2\gamma^\nu p_1]\langle p_1\gamma^\mu k_1]}{t}\right.\\
&\qquad\left.+\frac{\langle k_2\gamma^\mu k_1]\langle k_1\gamma^\nu k_1]-\langle k_2\gamma^\mu p_2]\langle p_2\gamma^\nu k_1]}{u}\right]\\
=&\frac{-2\mathrm{i}e^2}{u}\left[\frac{\langle k_1k_2\rangle[p_1k_1]\langle k_2p_2\rangle[k_1k_1]-\langle k_2p_1\rangle[k_1p_1]\langle k_2p_2\rangle[k_1p_1]}{t}\right.\\
&\qquad\left.+\frac{\langle k_2k_2\rangle[k_1p_1]\langle k_1p_2\rangle[k_1k_1]-\langle k_2k_2\rangle[p_2p_1]\langle p_2p_2\rangle[k_1k_1]}{u}\right]\\
=&2\mathrm{i}e^2\frac{\langle k_2p_1\rangle[k_1p_1]\langle k_2p_2\rangle[k_1p_1]}{tu},
\end{aligned}\tag{5.97}$$

where we have used the spin sum identity $\not{p} = p]\langle p + p\rangle[p$ in the third equality, and also the Fierz transformations. Note that all spinor products like $\langle pp\rangle$ and $[pp]$, or $\langle p\gamma^\mu k\rangle$ and $[p\gamma^\mu k]$ vanish. Square this amplitude, we get

$$\left|\mathrm{i}\mathcal{M}(e_R^+ e_L^- \to \gamma_L\gamma_R)\right|^2 = 4e^4\frac{t}{u}. \tag{5.98}$$

In the same way, we calculate other polarized amplitudes,

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_R^+ e_L^- \to \gamma_R\gamma_L)\\
=&-\mathrm{i}e^2\frac{[k_1\gamma_\mu p_1\rangle\langle k_2\gamma_\nu p_2]}{4\sqrt{(k_1\cdot p_1)(k_2\cdot p_2)}}\left[\frac{[k_2\gamma^\nu(\not{k}_1-\not{p}_1)\gamma^\mu k_1\rangle}{t}+\frac{[k_2\gamma^\mu(\not{k}_1-\not{p}_2)\gamma^\nu k_1\rangle}{u}\right]\\
=&\ 2\mathrm{i}e^2\frac{\langle k_2p_1\rangle[k_1p_2]\langle k_2p_2\rangle[k_1p_2]}{tu}.
\end{aligned} \tag{5.99}$$

Note that we have used a different set of auxiliary momenta in photons' polarizations. After evaluating the rest two nonvanishing amplitudes, we get the squared polarized amplitudes, as follows:

$$\left|\mathcal{M}(e_R^+ e_L^- \to \gamma_L\gamma_R)\right|^2 = \left|\mathcal{M}(e_L^+ e_R^- \to \gamma_R\gamma_L)\right|^2 = 4e^4\frac{t}{u}, \tag{5.100}$$

$$\left|\mathcal{M}(e_L^+ e_R^- \to \gamma_L\gamma_R)\right|^2 = \left|\mathcal{M}(e_L^+ e_R^- \to \gamma_L\gamma_R)\right|^2 = 4e^4\frac{u}{t}. \tag{5.101}$$

Then the differential cross section follows straightforwardly,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{16\pi s}\left(\frac{1}{4}\sum_{\text{spin}}|\mathrm{i}\mathcal{M}|^2\right) = \frac{2\pi\alpha^2}{s}\left(\frac{t}{u}+\frac{u}{t}\right), \tag{5.102}$$

which is in accordance with (5.107) of P&S.

Chapter 6

Radiative Corrections: Introduction

6.1 Rosenbluth formula

In this problem we derive the differential cross section for the electron-proton scattering in the lab frame, assuming that the scattering energy is much higher than electron's mass, and taking account of the form factors of the proton. The result is known as Rosenbluth formula. The relevant diagram is shown in Figure 6.1.

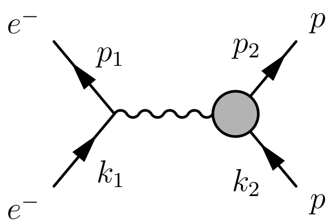


Figure 6.1: The electron-proton scattering. The blob denotes form factors that includes the effect of strong interaction. All initial momenta go inward and all final momenta go outward.

Let us firstly work out the kinematics. In the lab frame, the momenta can be parameterized as

$$k_1 = (E, 0, 0, E), \qquad p_1 = (E', E'\sin\theta, 0, E'\cos\theta), \qquad k_2 = (M, 0, 0, 0), \tag{6.1}$$

and p_2 can be found by momentum conservation, $k_1 + k_2 = p_1 + p_2$. With the on-shell condition $p_2^2 = M^2$, we find that

$$E' = \frac{ME}{M + 2E\sin^2\frac{\theta}{2}}. \tag{6.2}$$

We also use $q = k_1 - p_1$ to denote the momentum transfer and $t = q^2$ its square. Note that we have set the electron mass to zero.

Now we write down the amplitude $\mathcal{M}$.

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2 \bar{U}(p_2)\left[\gamma^\mu F_1(q^2) + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right]U(k_2)\frac{-\mathrm{i}}{t}\bar{u}(p_1)\gamma_\mu u(k_1), \tag{6.3}$$

where U is the spinor for the proton and u is for the electron, M is the mass of the proton. At this stage, we convert this expression into a more convenient form by means of the Gordon identity (see Problem 3.2),

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2 \bar{U}(p_2)\left[\gamma^\mu(F_1+F_2) - \frac{(p_2+k_2)^\mu}{2M}F_2\right]U(k_2)\frac{-\mathrm{i}}{t}\bar{u}(p_1)\gamma_\mu u(k_1). \tag{6.4}$$

Now, the squared amplitude with initial spins averaged and final spins summed is,

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{e^4}{4q^4}\,\mathrm{tr}\left[\left(\gamma^\mu(F_1+F_2) - \frac{(p_2+k_2)^\mu}{2M}F_2\right)(\not{k}_2+M)\right.\\ &\qquad\left.\times\left(\gamma^\rho(F_1+F_2) - \frac{(p_2+k_2)^\rho}{2M}F_2\right)(\not{p}_2+M)\right]\mathrm{tr}\left[\gamma_\mu \not{k}_1\gamma_\rho\not{p}_1\right]\\ &= \frac{4e^4M^2}{q^4}\left[\left(2E^2+2E'^2+q^2\right)(F_1+F_2)^2\right.\\ &\qquad\left. - \left(2F_1F_2 + F_2^2\left(1+\tfrac{q^2}{4M^2}\right)\right)\left((E+E')^2 + q^2\left(1-\tfrac{q^2}{4M^2}\right)\right)\right]. \end{aligned} \tag{6.5}$$

There are two terms in the square bracket in the last expression. We rewrite the first factor in the second term as

$$2F_1F_2 + F_2^2\left(1+\tfrac{q^2}{4M^2}\right) = (F_1+F_2)^2 - F_1^2 + \tfrac{q^2}{4M^2}F_2^2,$$

and combine the $(F_1+F_2)^2$ part into the first term, which leads to,

$$\frac{1}{4}\sum|\mathcal{M}|^2 = \frac{4e^4M^2}{q^4}\left[\tfrac{q^4}{2M^2}(F_1+F_2)^2 + 4\left(F_1^2 - \tfrac{q^2}{4M^2}F_2^2\right)EE'\cos^2\tfrac{\theta}{2}\right],$$

where we have used the following two relations which can be easily justified,

$$E' - E = \tfrac{q^2}{2m}, \tag{6.6}$$

$$q^2 = -4E'E\sin^2\tfrac{\theta}{2}. \tag{6.7}$$

Now we can put the squared amplitude into its final form,

$$\begin{aligned}\frac{1}{4}\sum|\mathcal{M}|^2 &= \frac{16e^4E^2M^3}{q^4\left(M+2E\sin^2\frac{\theta}{2}\right)}\\ &\qquad\times\left[\left(F_1^2 - \tfrac{q^2}{4M^2}F_2^2\right)\cos^2\tfrac{\theta}{2} - \frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\tfrac{\theta}{2}\right].\end{aligned} \tag{6.8}$$

On the other hand, we can derive the $A+B\to 1+2$ differential cross section in the lab frame as

$$\mathrm{d}\sigma_{\mathrm{L}} = \frac{1}{2E_A 2E_B|\mathbf{v}_A - \mathbf{v}_B|}\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_1 2E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B). \tag{6.9}$$

In our case, $E_A = E$, $E_B = M$, $E_1 = E'$, and $|\mathbf{v}_A - \mathbf{v}_B| \simeq 1$, thus,

$$\mathrm{d}\sigma_{\mathrm{L}} = \frac{1}{4EM}\int\frac{\mathrm{d}^3p_1\mathrm{d}^3p_2}{(2\pi)^6 2E_1 2E_2}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p_1+p_2-p_A-p_B)$$

$$
\begin{aligned}
&= \frac{1}{4EM}\int \frac{E'^2\mathrm{d}E'\mathrm{d}\cos\theta\mathrm{d}\varphi}{(2\pi)^3 2E'2E_2}|\mathcal{M}|^2(2\pi)\delta\big(E'+E_2(E')-E-M\big)\\
&= \frac{1}{4EM}\int \frac{E'^2\mathrm{d}E'\mathrm{d}\cos\theta\mathrm{d}\varphi}{(2\pi)^2 2E'2E_2}|\mathcal{M}|^2\left[1+\frac{E'-E\cos\theta}{E_2(E')}\right]^{-1}\delta\bigg(E'-\frac{ME}{M+2E\sin^2\frac{\theta}{2}}\bigg)\\
&= \frac{1}{4EM}\int \frac{\mathrm{d}\cos\theta}{8\pi}|\mathcal{M}|^2\frac{E'}{M+2E\sin^2\frac{\theta}{2}},
\end{aligned}
$$

where we use the notation $E_2 = E_2(E')$ to emphasize that E_2 is a function of E'. That is,

$$E_2 = \sqrt{M^2+E^2+E'^2-2E'E\cos\theta}.$$

Then,

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\mathrm{L}} = \frac{1}{32\pi\big(M+2E\sin^2\frac{\theta}{2}\big)^2}|\mathcal{M}|^2. \tag{6.10}$$

So finally we get the differential cross section, the Rosenbluth formula,

$$
\begin{aligned}
\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\Big)_{\mathrm{L}} = &\frac{\pi\alpha^2}{2E^2\big(1+\frac{2E}{M}\sin^2\frac{\theta}{2}\big)\sin^4\frac{\theta}{2}}\\
&\times\left[\big(F_1^2-\tfrac{q^2}{4M^2}F_2^2\big)\cos^2\tfrac{\theta}{2}-\frac{q^2}{2M^2}(F_1+F_2)^2\sin^2\tfrac{\theta}{2}\right].
\end{aligned}\tag{6.11}
$$

6.2 Equivalent photon approximation

In this problem we study the scattering of a very high energy electron from a target in the forward scattering limit. The relevant matrix element is,

$$\mathcal{M} = (-\mathrm{i}e)\bar{u}(p')\gamma^\mu u(p)\frac{-\mathrm{i}g_{\mu\nu}}{q^2}\widehat{\mathcal{M}}^\nu(q). \tag{6.12}$$

(a) First, the spinor product in the expression above can be expanded as,

$$\bar{u}(p')\gamma^\mu u(p) = A\cdot q^\mu + B\cdot\tilde{q}^\mu + C\cdot\epsilon_1^\mu + D\cdot\epsilon_2^\mu. \tag{6.13}$$

Now, using the fact that $q_\mu u(p')\gamma^\mu u(p) = 0$, we have,

$$0 = Aq^2 + Bq\cdot\tilde{q} \simeq -4AEE'\sin^2\tfrac{\theta}{2} + Bq\cdot\tilde{q} \qquad\Rightarrow\qquad B\sim\theta^2. \tag{6.14}$$

(b) It is easy to find that

$$\epsilon_1^\mu = N(0, p'\cos\theta - p, 0, -p'\sin\theta), \qquad\qquad \epsilon_2^\mu = (0,0,1,0),$$

where $N = (E^2+E'^2-2EE'\cos\theta)^{-1/2}$ is the normalization constant. Then, for the right-handed electron with spinor $u_+(p) = \sqrt{2E}(0,0,1,0)^T$ and left-handed electron with $u_-(p) = \sqrt{2E}(0,1,0,0)^T$, it is straightforward to show that

$$u_+(p') = \sqrt{2E'}(0,0,\cos\tfrac{\theta}{2},\sin\tfrac{\theta}{2})^T, \qquad u_-(p') = \sqrt{2E'}(-\sin\tfrac{\theta}{2},\cos\tfrac{\theta}{2},0,0), \tag{6.15}$$

and,

$$\bar{u}_\pm(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_1 u_\pm(p) \simeq -\sqrt{EE'}\frac{E+E'}{|E-E'|}\theta, \tag{6.16}$$

$$\bar{u}_\pm(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_2 u_\pm(p) \simeq \pm\mathrm{i}\sqrt{EE'}\theta, \tag{6.17}$$

$$\bar{u}_\pm(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_1 u_\mp(p) = \bar{u}_\pm(p')\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_2 u_\mp(p) = 0. \tag{6.18}$$

That is to say, we have,

$$C_\pm = -\sqrt{EE'}\frac{E+E'}{|E-E'|}\theta, \qquad D_\pm = \pm\mathrm{i}\sqrt{EE'}\theta. \tag{6.19}$$

(c) The squared amplitude is given by,

$$|\mathcal{M}_{\pm\pm}|^2 = \frac{e^2}{(q^2)^2}\widehat{\mathcal{M}}_\mu(q)\widehat{\mathcal{M}}_\nu(q)\big(C_\pm\epsilon_1^\mu + D_\pm\epsilon_2^\mu\big)\big(C_\pm^*\epsilon_1^{\nu*} + D_\pm^*\epsilon_2^{\nu*}\big). \tag{6.20}$$

Averaging and summing over the initial and final spins of the electron respectively, we get,

$$\begin{aligned}\frac{1}{2}\sum|\mathcal{M}|^2 &= \frac{e^2}{2(q^2)^2}\widehat{\mathcal{M}}_\mu(q)\widehat{\mathcal{M}}_\nu(q)\Big[\big(|C_+|^2+|C_-|^2\big)\epsilon_1^\mu\epsilon_1^{\nu*} + \big(|D_+|^2+|D_-|^2\big)\epsilon_2^\mu\epsilon_2^{\nu*} \\ &\qquad + \big(C_+D_+^* + C_-D_-^*\big)\epsilon_1^\mu\epsilon_2^{\nu*} + \big(C_+^*D_+ + C_-^*D_-\big)\epsilon_2^\mu\epsilon_1^{\nu*}\Big] \\ &= \frac{e^2}{(q^2)^2}\widehat{\mathcal{M}}_\mu(q)\widehat{\mathcal{M}}_\nu(q)EE'\theta^2\Big[\Big(\frac{E+E'}{E-E'}\Big)^2\epsilon_1^\mu\epsilon_1^{\nu*} + \epsilon_2^\mu\epsilon_2^{\nu*}\Big].\end{aligned} \tag{6.21}$$

Then the cross section reads,

$$\begin{aligned}\int\mathrm{d}\sigma &= \frac{1}{2E2M_\mathrm{t}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}\frac{\mathrm{d}^3p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}\Big(\frac{1}{2}\sum|\mathcal{M}|^2\Big)(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big) \\ &= \frac{e^2}{2E2M_\mathrm{t}}\int\frac{\mathrm{d}^3p'}{(2\pi)^3 2E'}\frac{EE'\theta^2}{3(q^2)^2}\Big[\Big(\frac{E+E'}{E-E'}\Big)^2+1\Big] \\ &\quad\times\int\frac{\mathrm{d}^3p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big) \\ &= -\frac{1}{2E2M_\mathrm{t}}\frac{\alpha}{2\pi}\int\mathrm{d}x\Big[1+\Big(\frac{2-x}{x}\Big)^2\Big]\int_0^\pi\mathrm{d}\theta\frac{\theta^2\sin\theta}{4(1-\cos\theta)^2} \\ &\quad\times\int\frac{\mathrm{d}^3p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big).\end{aligned} \tag{6.22}$$

where we have used the trick described in the final project of Part I (radiation of gluon jets) to separate the contractions of Lorentz indices, and $x \equiv (E-E')/E$. Now let us focus on the integral over the scattering angle θ in the last expression, which is contributed from the following factor,

$$\int_0^\pi\mathrm{d}\theta\,\frac{\theta^2\sin\theta}{4(1-\cos\theta)^4} \sim \int_0\frac{\mathrm{d}\theta}{\theta}. \tag{6.23}$$

which is logarithmically divergent as $\theta\to 0$.

(d) We reintroduce the mass of the electron into the denominator to cut off the divergence, namely, let $q^2 = -2(EE' - pp'\cos\theta) + 2m^2$. Then we can expand q^2, treating m^2 and θ as small quantities, as,

$$q^2 \simeq -(1-x)E^2\theta^2 - \frac{x^2}{1-x}m^2.$$

Then the polar angle integration near $\theta = 0$ becomes,

$$\int_0 \mathrm{d}\theta\, \theta^3 \Big[\theta^2 + \frac{x^2}{(1-x)^2}\frac{m^2}{E^2}\Big]^{-2} \sim -\frac{1}{2}\log\frac{E^2}{m^2}. \tag{6.24}$$

(e) Combining the results above, the cross section can be expressed as,

$$\begin{aligned}\int \mathrm{d}\sigma =& -\frac{1}{2E2M_\mathrm{t}}\frac{\alpha}{2\pi}\int \mathrm{d}x \Big[1+\Big(\frac{2-x}{x}\Big)^2\Big]\int_0 \mathrm{d}\theta\,\theta^3\Big[\theta^2 + \frac{x^2}{(1-x)^2}\frac{m^2}{E^2}\Big]^{-2} \\ &\times \int \frac{\mathrm{d}^3 p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big) \\ =& \frac{1}{2E2M_\mathrm{t}}\frac{\alpha}{2\pi}\int \mathrm{d}x\,\frac{1+(1-x)^2}{x^2}\log\frac{E^2}{m^2} \\ &\times \int \frac{\mathrm{d}^3 p_\mathrm{t}}{(2\pi^3)2E_\mathrm{t}}|\widehat{\mathcal{M}}_\mu(q)|^2(2\pi)^4\delta^{(4)}\big(\textstyle\sum p_i\big).\end{aligned} \tag{6.25}$$

6.3 Exotic contributions to $g-2$

(a) The 1-loop vertex correction from Higgs boson is,

$$\begin{aligned}\bar{u}(p')\delta\Gamma^\mu u(p) =& \Big(\frac{\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_h^2}\bar{u}(p')\frac{\mathrm{i}}{\not{k}+\not{q}-m}\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}u(p) \\ =& \frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^d k'}{(2\pi)^d}\frac{2\bar{u}(p')N^\mu u(p)}{(k'^2-\Delta)^3},\end{aligned} \tag{6.26}$$

with

$$N^\mu = (\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m), \tag{6.27}$$

$$k' = k - xp + yq, \tag{6.28}$$

$$\Delta = (1-x)m^2 + xm_h^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q. \tag{6.29}$$

To put this correction into the following form,

$$\Gamma^\mu = \gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q), \tag{6.30}$$

we first rewrite N^μ as,

$$N^\mu = A\gamma^\mu + B(p'+p)^\mu + C(p'-p)^\mu, \tag{6.31}$$

where term proportional to $(p'-p)$ can be thrown away by Ward identity $q_\mu\Gamma^\mu(q) = 0$. This can be done by gamma matrix calculations, leading to the following result,

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (3+2x-x^2)m^2 + (y-xy-y^2)q^2\Big]\gamma^\mu + (x^2-1)m(p'+p)^\mu. \tag{6.32}$$

Then, using Gordon identity, we find,

$$N^\mu = \Big[\big(\tfrac{2}{d}-1\big)k'^2 + (x+1)^2m^2 + (y-y^2-xy)q^2\Big]\gamma^\mu + \frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}\cdot 2m^2(1-x^2). \tag{6.33}$$

Comparing this with (6.30), we see that

$$\begin{aligned}\delta F_2(q=0) &= 2\mathrm{i}\lambda^2m^2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{1-x^2}{(k'^2-\Delta)^3}\\ &= \frac{\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}x\,\frac{(1-x)^2(1+x)}{(1-x)^2+x(m_h/m)^2}.\end{aligned} \tag{6.34}$$

To carry out the integration over x, we use the approximation that $m_h \gg m$. Then,

$$\begin{aligned}\delta F_2(q=0) &\simeq \frac{\lambda^2}{(4\pi)^2}\int_0^1\mathrm{d}x\left[\frac{1}{1+x(m_h/m)^2}-\frac{1+x-x^2}{(m_h/m)^2}\right]\\ &\simeq \frac{\lambda^2}{(4\pi)^2(m_h/m)^2}\Big[\log(m_h^2/m^2)-\tfrac{7}{6}\Big].\end{aligned} \tag{6.35}$$

(b) According to (a), the limits on λ and m_h is given by,

$$\delta F_2(q=0) = \frac{\lambda^2}{(4\pi)^2(m_h/m)^2}\Big[\log(m_h^2/m^2)-\tfrac{7}{6}\Big] < 1\times 10^{-10}. \tag{6.36}$$

For electron, $\lambda \simeq 3\times 10^{-6}$, $m \simeq 0.511\text{MeV}$, and with $m_h \sim 60\,\text{GeV}$, we have $\delta F_2(q=0) \sim 10^{-22} \ll 10^{-10}$. For realistic case $m_h \simeq 125\text{GeV}$ the effect is even smaller. On the other hand, for muon, we have $\lambda = 6\times 10^{-4}$, $m \simeq 106\text{MeV}$, and with the input $m_h \simeq 125\text{GeV}$, we have $\delta F_2(q=0) \sim 10^{-14}$. At present the experimentally measured muon's anomalous magnetic moment is a bit different from the prediction of Standard Model, and the difference is of order 10^{-9}, a not decisive but still noteworthy "anomaly". More can be found in [3].

(c) The 1-loop correction from the axion is given by,

$$\begin{aligned}\bar u(p')\delta\Gamma^\mu u(p) =& \Big(\frac{-\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_h^2}\bar u(p')\gamma^5\frac{\mathrm{i}}{\not k+\not q-m}\gamma^\mu\frac{\mathrm{i}}{\not k-m}\gamma^5u(p)\\ =& -\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{2\bar u(p')N^\mu u(p)}{(k'^2-\Delta)^3},\end{aligned} \tag{6.37}$$

in which k' and Δ are still defined as in (a) except the replacement $m_h \to m_a$, while N^μ is now given by,

$$N^\mu = \gamma^5(\not k+\not q+m)\gamma^\mu(\not k+m)\gamma^5 = -(\not k+\not q-m)\gamma^\mu(\not k-m). \tag{6.38}$$

Repeating the same derivation as was done in (a), we get,

$$N^\mu = \big[-\big(\tfrac{2}{d}-1\big)k'^2-(1-x-y)yq^2+(1-x)^2m^2\big]\gamma^\mu-(1-x)^2m(p'+p)^2. \tag{6.39}$$

Again, using Gordon identity, we get,

$$
\begin{aligned}
\delta F_2(q=0) &= -2\mathrm{i}\lambda^2 m^2 \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{(1-x)^2}{(k'^2-\Delta)^3} \\
&= -\frac{\lambda^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \, \frac{(1-x)^3}{(1-x)^2 + x m_a^2/m^2} \\
&\simeq -\frac{\lambda^2}{(4\pi)^2} \int_0^1 \mathrm{d}x \left[\frac{1}{1 + x m_a^2/m^2} - \frac{3 - 3x + x^2}{m_a^2/m^2} \right] \\
&= -\frac{\lambda^2}{(4\pi)^2 (m_a^2/m^2)} \left[\log(m_a^2/m^2) - \tfrac{11}{6} \right].
\end{aligned} \tag{6.40}
$$

For order-of-magnitude estimation, it's easy to see that $\lambda m/m_a \gtrsim 10^{-5}$ is excluded.

Chapter 7

Radiative Corrections: Some Formal Developments

7.1 Optical theorem in ϕ^4 theory

In this problem we check the optical theorem in phi-4 theory to order λ^2. Firstly, the total cross section σ_{tot} at this order receives contributions from tree level only. The squared amplitude is simply λ^2. Then its easy to get the total cross section by complementing kinematic factors. That is,

$$\sigma_{\text{tot}} = \frac{\lambda^2}{16\pi s}, \tag{7.1}$$

where $s = E_{\text{CM}}^2$ and E_{CM} is the COM energy. Then, consider the imaginary part of the scattering amplitude. The contribution comes from 1-loop diagram in s-channel this time. Let's evaluate this amplitude directly,

$$\begin{aligned} \mathrm{i}\mathcal{M} &= \frac{1}{2}(-\mathrm{i}\lambda)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \frac{\mathrm{i}}{(k-p)^2 - m^2} = \frac{\lambda^2}{2} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \, \frac{1}{(k'^2 - \Delta)^2} \\ &= \frac{\mathrm{i}\lambda^2}{2(4\pi)^2} \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \int_0^1 \mathrm{d}x \, \log\big(m^2 - x(1-x)s\big) \Big]. \end{aligned} \tag{7.2}$$

Therefore,

$$\mathrm{Im}\mathcal{M} = -\frac{\lambda^2}{2(4\pi)^2} \int_0^1 \mathrm{d}x \, \mathrm{Im}\Big[\log\big(m^2 - x(1-x)s\big) \Big]. \tag{7.3}$$

The argument in the logarithm is real, thus the imaginary part of the logarithm equals to 0 or $-\pi$ depending on the argument is positive or negative. (Strictly speaking the imaginary part is $-\pi$ but not π due to our iϵ prescription.) Then we see this logarithm contributes an constant imaginary part $-\pi$, only when

$$\frac{1 - \sqrt{1 - 4m^2/s}}{2} < x < \frac{1 + \sqrt{1 - 4m^2/s}}{2}.$$

Thus we have

$$\mathrm{Im}\mathcal{M} = \frac{\lambda^2}{32\pi} \sqrt{1 - 4m^2/s} = \frac{\lambda^2 p_{\text{CM}}}{16\pi E_{\text{CM}}}. \tag{7.4}$$

7.2 Alternative regulators in QED

In this problem we compute the first order corrections to Z_1 and Z_2 in QED, using cut-off regularization and dimensional regularization respectively. By definition, we have,

$$\Gamma^\mu(q=0) = Z_1^{-1}\gamma^\mu, \tag{7.5}$$

$$Z_2^{-1} = 1 - \frac{\mathrm{d}\Sigma}{\mathrm{d}\not{p}}\bigg|_{\not{p}=m}. \tag{7.6}$$

Let's begin with dimensional regularization instead of momentum cut-off.

(b) Dimensional Regularization We firstly calculate $\delta F_1(0)$:

$$\begin{aligned}
&\bar{u}(p')\delta\Gamma^\mu(p,p')u(p)\\
=&(-ie)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-ig_{\rho\sigma}}{(k-p)^2-\mu^2}\bar{u}(p')\gamma^\sigma\frac{i}{\not{k}+\not{q}-m}\gamma^\mu\frac{i}{\not{k}-m}\gamma^\rho u(p)\\
=&-\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\bar{u}(p')\gamma_\rho(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma^\rho u(p)}{\big((k-p)^2-\mu^2\big)\big((k+q)^2-m^2\big)\big(k^2-m^2\big)}\\
=&-\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{2\bar{u}(p')N^\mu u(p)}{(k'^2-\Delta)^3},
\end{aligned}\tag{7.7}$$

in which we define

$$\begin{aligned}
&k' = k - xp + yq,\\
&\Delta = (1-x)m^2 + x\mu^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q,\\
&N^\mu = \gamma_\rho(\not{k}+\not{q}+m)\gamma^\mu(\not{k}+m)\gamma^\rho.
\end{aligned}$$

The next step is to put N^μ into the needed form. The calculation is basically in parallel with Peskin's Sec.6.3. Let me show some details. The first step is to finish the summation over dummy Lorentz indices. Note that we are using dimensional regularization, thus we should use Peskin's Eq.(A.55). The result is:

$$N^\mu = -2\not{k}\gamma^\mu(\not{k}+\not{q}) + 4m(2k+q)^\mu - (d-2)m^2\gamma^\mu + (4-d)\Big[(\not{k}+\not{q})\gamma^\mu\not{k} - m(\not{k}+\not{q})\gamma^\mu - m\gamma^\mu\not{k}\Big].$$

It's worth noting here that d will be sent to 4 at the end of the calculation. Thus in the square bracket in this expression, only the combination $\not{k}\gamma^\mu\not{k}$ contributes to the final result. Thus we simply have

$$N^\mu = -2\not{k}\gamma^\mu(\not{k}+\not{q}) + 4m(2k+q)^\mu - (d-2)m^2\gamma^\mu + (4-d)\not{k}\gamma^\mu\not{k}. \tag{7.8}$$

Here and following, we are free to drop off terms in N^μ which contribute nothing to final results. The equal sign should be understood in this way. The next step is to rewrite N^μ in terms of k' instead of k:

$$N^\mu = (2-d)\not{k}'\gamma^\mu\not{k}' - 2\big[x\not{p} - y\not{q}\big]\gamma^\mu\big[x\not{p} + (1-y)\not{q}\big] + 4m\big[2xp + (1-2y)q\big]^\mu - 2m^2\gamma^\mu. \tag{7.9}$$

Terms linear in k' has been dropped since they integrate to zero. The third step is to put N^μ into a linear combination of γ^μ, $(p+p')^\mu$. Terms proportional to $(p-p')^\mu$ will be dropped due to Ward identity. The basic strategy is using substitution $q^\mu = (p'-p)^\mu$, on shell condition $\bar{u}(p')\not{p}' = \bar{u}(p')m$ and $\not{p}u(p) = mu(p)$. Here we show the detailed steps for the second term above:

$$\begin{aligned}
&-2\big[x\not{p} - y\not{q}\big]\gamma^\mu\big[x\not{p} + (1-y)\not{q}\big] \\
=&-2\big[x^2\not{p}\gamma^\mu\not{p} - y(1-y)\not{q}\gamma^\mu\not{q} - xy\not{q}\gamma^\mu\not{p} + x(1-y)\not{p}\gamma^\mu\not{q}\big] \\
=&-2\big[x^2(2p^\mu - \gamma^\mu\not{p})\not{p} - y(1-y)(2q^\mu - \gamma^\mu\not{q})\not{q} - xy\not{q}\gamma^\mu\not{p} + x(1-y)(\not{p}' - \not{q})\gamma^\mu\not{q}\big] \\
=&-2\big[2x^2mp^\mu - x^2m^2\gamma^\mu + y(1-y)q^2\gamma^\mu - 2xymq^\mu + xm\gamma^\mu\not{q} - x(1-y)\not{q}\gamma^\mu\not{q}\big] \\
=&-2\big[-x(x+2)m^2\gamma^\mu + (x+y)(1-y)q^2\gamma^\mu + 2x^2mp^\mu + 2xmp'^\mu\big].
\end{aligned}$$

Combining this with other terms, and also make the momentum symmetrization (Peskin's Eq.(7.87)), which amounts to make the substitution:

$$\not{k}'\gamma^\mu\not{k}' \to \big(\tfrac{2}{d} - 1\big)k'^2\gamma^\mu,$$

we get

$$N^\mu = \tfrac{(2-d)^2}{d}\not{k}'\gamma^\mu\not{k}' + \big[2(x^2+2x-1)m^2 - 2(x+y)(1-y)q^2\big]\gamma^\mu + 2x(1-x)m(p'+p)^\mu. \quad (7.10)$$

Now we employ Gordon's identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\Big[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\Big]u(p),$$

to put N^μ into a linear combination of γ^μ and $\sigma^{\mu\nu}$:

$$N^\mu = \big[\tfrac{(2-d)^2}{d}k'^2 - 2(x^2-4x+1)m^2 - 2(x+y)(1-y)q^2\big]\gamma^\mu - 2x(1-x)mi\sigma^{\mu\nu}q_\nu. \quad (7.11)$$

Now we have put the vertex Γ^μ into the following form:

$$\Gamma^\mu = \gamma^\mu F_1(q) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q). \quad (7.12)$$

We are interest in $\delta F_1(q)$, which is related to δZ_1 by $\delta Z_1 = -\delta F_1(q=0)$. Finishing momentum integral:

$$\begin{aligned}
\delta F_1(0) =& -2\mathrm{i}e^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\frac{1}{(k'^2-\Delta)^3}\big[\tfrac{(2-d)^2}{d}k'^2 - 2(x^2-4x+1)m^2\big] \\
=&\frac{2e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\Big[\frac{(2-d)^2\Gamma(2-\frac{d}{2})}{4\Delta^{2-d/2}} + (x^2-4x+1)m^2\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}\Big], \quad (7.13)
\end{aligned}$$

and sending $d = 4-\epsilon \to 4$:

$$\delta F_1(0) = \frac{2e^2}{(4\pi)^2}\int_0^1\mathrm{d}x(1-x)\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi$$

$$-\log\big((1-x)^2m^2+x\mu^2\big)-2+\frac{(x^2-4x+1)m^2}{(1-x)^2m^2+x\mu^2}\Big], \tag{7.14}$$

we reach the needed result $\delta Z_1=-\delta F_1(0)$. Now let us turn to Z_2. The correction of first order is given by $\delta Z_2=(\mathrm{d}\Sigma/\mathrm{d}\not{p})\big|_{\not{p}=m}$. Therefore we'd better evaluate $\Sigma(\not{p})$ using dimensional regularization.

$$\begin{aligned}
-\mathrm{i}\Sigma(\not{p}) =&(-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}\gamma_\mu\frac{-\mathrm{i}}{(p-k)^2-\mu^2}\\
=&-e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(2-d)\not{k}+dm}{(k^2-m^2)\big((p-k)^2-\mu^2\big)}=-e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\frac{(2-d)x\not{p}+dm}{(k'^2-\Delta)^2}\\
=&-\frac{\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[(2-d)x\not{p}+dm\big],
\end{aligned}\tag{7.15}$$

where

$$\begin{aligned}
k'&=k-xp;\\
\Delta&=(1-x)m^2+x\mu^2-x(1-x)p^2.
\end{aligned}$$

Then we can compute

$$\begin{aligned}
\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}=&\frac{e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\Big[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(2-d)x-\frac{(2-\frac{d}{2})\Gamma(2-\frac{d}{2})}{\Delta^{3-d/2}}\frac{\mathrm{d}\Delta}{\mathrm{d}\not{p}}\big((2-d)x\not{p}+dm\big)\Big]\\
=&\frac{e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\Big[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(2-d)x+\frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}}2x(1-x)\not{p}\big((2-d)x\not{p}+dm\big)\Big].
\end{aligned}\tag{7.16}$$

Then, setting $\not{p}=m$ and $d=4-\epsilon$ with $\epsilon\to0$, we get

$$\begin{aligned}
\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m}=&\frac{-2e^2}{(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[\tfrac{2}{\epsilon}+\gamma+\log4\pi\\
&-\log\big((1-x)^2m^2+x\mu^2\big)-1-\frac{2(1-x)(2-x)m^2}{(1-x)^2m^2+x\mu^2}\Big]
\end{aligned}\tag{7.17}$$

Now it's still not easy to see $\delta Z_1=\delta Z_2$ immediately. To make this transparent, we need some more steps. Let's focus on logarithm term:

$$\begin{aligned}
&-\int_0^1\mathrm{d}x(1-x)\log\big((1-x)^2m^2+x\mu^2\big)=-\int_0^1\mathrm{d}x(1-2x+x)\log\big((1-x)^2m^2+x\mu^2\big)\\
=&\int_0^1\mathrm{d}x\Big[(1-x)-\frac{(1-x)(1-x^2)m^2}{(1-x^2)m^2+x\mu^2}-x\log\big((1-x)^2m^2+x\mu^2\big)\Big]
\end{aligned}\tag{7.18}$$

Combining this with other terms, and also using the fact $\int x\mathrm{d}x=\int(1-x)\mathrm{d}x$, we get

$$\begin{aligned}
\delta F_1(0)=&\frac{2e^2}{(4\pi)^2}\int_0^1\mathrm{d}xx\Big[\tfrac{2}{\epsilon}-\gamma+\log4\pi\\
&-\log\big((1-x)^2m^2+x\mu^2\big)-1-\frac{2(1-x)(2-x)m^2}{(1-x)^2m^2+x\mu^2}\Big].
\end{aligned}\tag{7.19}$$

Now it's clear that $\delta Z_1=\delta Z_2$. Thus $Z_1=Z_2$ keeps unaffected at this order when dimensional regularization is used.

(a) Momentum Cut-off Now we repeat the calculation above using momentum cut-off. Now we can directly borrow some results above. All we need to do is setting $d \to 4$ and adding a UV momentum cut-off Λ, as well as the following integral formulae:

$$\int^\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^2} = \frac{\mathrm{i}}{16\pi^2}\Big[\log\big(1+\tfrac{\Lambda^2}{\Delta}\big) - \tfrac{\Lambda^2}{\Lambda^2+\Delta}\Big],$$
$$\int^\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{k^2}{(k^2-\Delta)^3} = \frac{\mathrm{i}}{16\pi^2}\Big[\log\big(1+\tfrac{\Lambda^2}{\Delta}\big) + \tfrac{\Delta(4\Lambda^2+3\Delta)}{2(\Lambda^2+\Delta)^2} - \tfrac{3}{2}\Big],$$
$$\int^\Lambda \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{1}{(k^2-\Delta)^3} = -\frac{\mathrm{i}}{32\pi^2}\frac{\Lambda^4}{\Delta(\Lambda^2+\Delta)^2}.$$

We begin with (7.13):

$$\begin{aligned}\delta F_1(0) =& -2\mathrm{i}e^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int^\Lambda \frac{\mathrm{d}^4 k'}{(2\pi)^4}\frac{1}{(k'^2-\Delta)^3}\left[k'^2 - 2(x^2-4x+1)m^2\right]\\ =& \frac{e^2}{8\pi^2}\int_0^1 \mathrm{d}x\,(1-x)\bigg[\log\Big(1+\frac{\Lambda^2}{\Delta}\Big) + \frac{(x^2-4x+1)m^2}{\Delta} - \frac{3}{2}\bigg],\end{aligned} \tag{7.20}$$

In the same way, we get

$$-\mathrm{i}\Sigma(\not{p}) = 2e^2\int^\Lambda \frac{\mathrm{d}^4 k'}{(2\pi)^4}\int_0^1 \mathrm{d}x\frac{x\not{p}-2m}{(k'^2-\Delta)^2}, \tag{7.21}$$

and

$$\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\bigg|_{\not{p}=m} = \frac{-e^2}{8\pi^2}\int_0^1 \mathrm{d}x\left[x\log\Big(1+\frac{\Lambda^2}{\Delta}\Big) - x + \frac{2x(1-x)(x-2)m^2}{\Delta}\right]. \tag{7.22}$$

This shows that $\delta Z_1 \neq \delta Z_2$ with cut-off regularization.

7.3 Radiative corrections in QED with Yukawa interaction

(a) Let us calculate the first order corrections to Z_1 and Z_2, as was done in Problem 7.2. Firstly, we calculate $\delta\Gamma^\mu$, which is similar to the corresponding QED correction:

$$\begin{aligned}&\bar{u}(p')\delta\Gamma^\mu(p,p')u(p)\\ =&(-\mathrm{i}\lambda^2/\sqrt{2})^2\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_\phi^2}\bar{u}(p')\frac{\mathrm{i}}{\not{k}+\not{q}-m}\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}u(p)\\ =&\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{2\bar{u}(p')N^\mu u(p)}{(k'^2-\Delta)^3},\end{aligned} \tag{7.23}$$

where

$$k' = k - xp + yq,$$

$$\Delta = (1-x)m^2 + xm_\phi^2 - x(1-x)p^2 - y(1-y)q^2 + 2xyp\cdot q,$$
$$N^\mu = (\not{k} + \not{q} + m)\gamma^\mu(\not{k} + m).$$

Then we put this correction into the following form, in parallel with steps of Problem 7.2. That is: (1) replace k by k' in N^μ:

$$N^\mu = \not{k}'\gamma^\mu\not{k}' + \big(x\not{p} + (1-y)\not{q} + m\big)\gamma^\mu(x\not{p} - y\not{q} + m);$$

(2) rewrite the numerator N^μ by gamma matrix relations and equations of motion, as

$$N^\mu = \Big[\big(\tfrac{2}{d} - 1\big)k'^2 + (3 + 2x - x^2)m^2 + (y - xy - y^2)q^2\Big]\gamma^\mu + (x^2 - 1)m(p' + p)^\mu;$$

(3) use Gordon identity to further transform N^μ into:

$$N^\mu = \Big[\big(\tfrac{2}{d} - 1\big)k'^2 + (x+1)^2m^2 + (y - y^2 - xy)q^2\Big]\gamma^\mu + \frac{\mathrm{i}\sigma^{\mu\nu}}{2m}\cdot 2m^2(1-x^2).$$

Then, we can read off δF_1 from the coefficient of γ^μ, as:

$$\begin{aligned}\delta F_1(0) &= \mathrm{i}\lambda^2\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k'^2-\Delta)^3}\Big[\big(\tfrac{2}{d}-1\big)k'^2 + (x+1)^2m^2\Big]\\ &= \frac{\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,(1-x)\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &\qquad - \log\big((1-x)^2m^2 + xm_\phi^2\big) - 1 + \frac{(x+1)^2m^2}{(1-x)^2m^2 + xm_\phi^2}\Big].\end{aligned} \tag{7.24}$$

Using the trick identity (7.18) again, we finally get

$$\begin{aligned}\delta F_1(0) = \frac{\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[&\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &- \log\big((1-x)^2m^2 + xm_\phi^2\big) + \frac{2(1-x^2)m^2}{(1-x)^2m^2 + xm_\phi^2}\Big].\end{aligned} \tag{7.25}$$

Now we calculate $\Sigma(\not{p})$.

$$\begin{aligned}-\mathrm{i}\Sigma(\not{p}) &= (-\mathrm{i}\lambda/\sqrt{2})^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not{k}-m}\frac{\mathrm{i}}{(p-k)^2-\mu^2} = \frac{\lambda^2}{2}\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{x\not{p}+m}{(k'^2-\Delta)^2}\\ &= \frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log\Delta\Big](x\not{p}+m),\end{aligned} \tag{7.26}$$

where $k' = k - xp$ and $\Delta = (1-x)m^2 + x\mu^2 - x(1-x)p^2$. Then we have

$$\begin{aligned}\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m} = \frac{-\lambda^2}{2(4\pi)^2}\int_0^1\mathrm{d}x\,x\Big[&\tfrac{2}{\epsilon} - \gamma + \log 4\pi\\ &- \log\big((1-x)^2m^2 + xm_\phi^2\big) + \frac{2(1-x^2)m^2}{(1-x)^2m^2 + xm_\phi^2}\Big].\end{aligned} \tag{7.27}$$

Thus we have proved that $\delta Z_1 = \delta Z_2$ holds for 1-loop scalar corrections.

(b) Now consider the 1-loop corrections to Yukawa vertex. We focus on the divergent part only. The equalities below should be understood to be hold up to a finite part. Then, for vertex correction from photon, we have

$$\begin{aligned}\delta\Gamma(p,p')\big|_{\text{photon}} =&(-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{(k-p)^2-\mu^2}\gamma^\mu\frac{\mathrm{i}}{\not{k}+\not{q}-m}\frac{\mathrm{i}}{\not{k}-m}\gamma_\mu\\ =&-\mathrm{i}e^2\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2dk'^2}{(k'^2-\Delta)^3}\\ =&\frac{d^2e^2}{2(4\pi)^{d/2}}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}=\frac{4e^2}{(4\pi)^2}\frac{2}{\epsilon}\end{aligned}\tag{7.28}$$

In the same way,

$$\begin{aligned}\delta\Gamma(p,p')\big|_{\text{scalar}} =&\Big(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{(k-p)^2-m_\phi^2}\frac{\mathrm{i}}{\not{k}+\not{q}-m}\frac{\mathrm{i}}{\not{k}-m}\\ =&\frac{\mathrm{i}\lambda^2}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2k'^2}{(k'^2-\Delta)^3}=\frac{-\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon}\end{aligned}\tag{7.29}$$

On the other hand, the 1-loop corrections for electron's self-energy also come from two parts: one is the photon correction, which has been evaluated in Problem 7.1,

$$-\mathrm{i}\Sigma(\not{p})\big|_{\text{photon}}=-\frac{\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[(2-d)x\not{p}+dm\big]=\frac{\mathrm{i}e^2(\not{p}-4m)}{(4\pi)^2}\frac{2}{\epsilon},\tag{7.30}$$

and the other is the scalar correction:

$$-\mathrm{i}\Sigma(\not{p})\big|_{\text{scalar}}=\Big(\frac{-\mathrm{i}\lambda}{\sqrt{2}}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not{k}-m}\frac{\mathrm{i}}{(p-k)^2-m_\phi^2}=\frac{\mathrm{i}\lambda^2(\not{p}+2m)}{4(4\pi)^2}\frac{2}{\epsilon},\tag{7.31}$$

To sum up, we have got the total vertex correction:

$$\delta\Gamma(p,p')=\delta\Gamma(p,p')\big|_{\text{photon}}+\delta\Gamma(p,p')\big|_{\text{scalar}}=\frac{4e^2-\lambda^2/2}{(4\pi)^2}\frac{2}{\epsilon},\tag{7.32}$$

and also:

$$\frac{\mathrm{d}\Sigma(\not{p})}{\mathrm{d}\not{p}}\Big|_{\not{p}=m}=\frac{\mathrm{d}\big[\Sigma(\not{p})_{\text{photon}}+\Sigma(\not{p})_{\text{scalar}}\big]}{\mathrm{d}\not{p}}\Big|_{\not{p}=m}=-\frac{e^2+\lambda^2/4}{(4\pi)^2}\frac{2}{\epsilon}.\tag{7.33}$$

Final Project I
Radiation of Gluon Jets

In this final project we do a basic exercise about cancellation of infrared divergence. An excellent pedagogical treatment of infrared divergence can be found in [2]

(a) First we calculate the 1-loop vertex correction to $\mathcal{M}(e^+e^- \to q\bar{q})$ from virtual gluon. The amplitude is given by

$$\mathrm{i}\delta_1\mathcal{M} = Q_f(-\mathrm{i}e)^2(-\mathrm{i}g)^2\bar{u}(p_1)\Big[\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^\nu\frac{\mathrm{i}}{\not{k}}\gamma^\mu\frac{\mathrm{i}}{\not{k}-\not{q}}\gamma_\nu\frac{-\mathrm{i}}{(k-p_1)^2-\mu^2}\Big]v(p_2)\frac{-\mathrm{i}}{q^2}\bar{v}(k_2)\gamma_\mu u(k_1). \tag{7.34}$$

Now we simplify the loop integral in the standard way, as was done in Problem 7.2. The result is

$$\begin{aligned}
\mathrm{i}\delta_1\mathcal{M} =& -\mathrm{i}g^2\Big[\int\frac{\mathrm{d}^dk}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2\big(\frac{(2-d)^2}{d}k'^2-2(1-x)(x+y)q^2\big)}{(k'^2-\Delta)^3}\Big]\mathrm{i}\mathcal{M}_0\\
=&\frac{2g^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\Big[\frac{(2-d)^2}{4\Delta^{2-d/2}}\Gamma(2-\tfrac{d}{2})+\frac{(1-x)(x+y)q^2}{\Delta^{3-d/2}}\Gamma(3-\tfrac{d}{2})\Big]\mathrm{i}\mathcal{M}_0\\
=&\frac{2g^2}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\Big[\tfrac{2}{\epsilon}-\gamma+\log 4\pi-\log\Delta-2+\frac{(1-x)(x+y)q^2}{\Delta}\Big]\mathrm{i}\mathcal{M}_0,
\end{aligned} \tag{7.35}$$

where

$$\mathrm{i}\mathcal{M}_0 = Q_f(-\mathrm{i}e)^2\bar{u}(p_1)\gamma^\mu\bar{v}(p_2)\frac{1}{q^2}v(k_2)\gamma_\mu u(k_1) \tag{7.36}$$

is the tree amplitude, and

$$k' = k - xq - yp_1, \qquad \Delta = -x(1-x-y)q^2 - y(1-y)p_1^2 + y\mu^2. \tag{7.37}$$

With the external legs amputated, the result is,

$$\mathrm{i}\delta_1\mathcal{M} = \frac{2g^2}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\Big[\log\Big(\frac{y\mu^2}{y\mu^2-x(1-x-y)q^2}\Big)+\frac{(1-x)(x+y)q^2}{y\mu^2-x(1-x-y)q^2}\Big]\mathrm{i}\mathcal{M}_0, \tag{7.38}$$

Then the cross section is given by

$$\sigma(e^+e^- \to q\bar{q}) = \frac{4\pi\alpha^2}{3s} \cdot 3|F_1(q^2 = s)|^2, \tag{7.39}$$

with

$$F_1(q^2 = s) = Q_f^2 + \frac{Q_f^2 \alpha_g}{2\pi} \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y \Big[\log\Big(\frac{y\mu^2}{y\mu^2 - x(1-x-y)s}\Big) + \frac{(1-x)(x+y)s}{y\mu^2 - x(1-x-y)s}\Big]. \tag{7.40}$$

We will carry out the Feynman integration in (e).

(b) Now we simplify the 3-body phase space integral

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}^3k_1 \mathrm{d}^3k_2 \mathrm{d}^3k_3}{(2\pi)^9 2E_1 2E_2 2E_3} \delta^{(4)}(q - k_1 - k_2 - k_3) \tag{7.41}$$

in the center of mass frame. It is convenient to introduce a new set of variables $x_i = 2k_i \cdot q/q^2$, $(i = 1, 2, 3)$. In COM frame, $x_i = 2E_i/E_q$ Then one can show that all Lorentz scalars involving final states only can be represented in terms of x_i and particles masses. In fact, we only need to check $(k_1 + k_2)^2$, $(k_2 + k_3)^2$ and $(k_3 + k_1)^2$. From instance,

$$(k_1 + k_2)^2 = (q - k_3)^2 = q^2 + m_3^2 - 2q \cdot k_3 = s(1 - x_3) + m_3^2. \tag{7.42}$$

Similarly,

$$(k_2 + k_3)^2 = s(1 - x_1) + m_1^2, \qquad (k_3 + k_1)^2 = s(1 - x_2) + m_2^2. \tag{7.43}$$

To simply the phase integral, we first integrate out k_3 with spatial delta function that restricts $k_3 = k_1 + k_2$:

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}^3k_1 \mathrm{d}^3k_2}{(2\pi)^6 2E_1 2E_2 2E_3} (2\pi)\delta(E_q - E_1 - E_2 - E_3). \tag{7.44}$$

The integral measure can be rewritten as

$$\mathrm{d}^3k_1 \mathrm{d}^3k_2 = k_1^2 k_2^2 \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}\Omega_1 \mathrm{d}\Omega_{12}, \tag{7.45}$$

where $\mathrm{d}\Omega_1$ is the spherical integral measure associated with d^3k_1, and $\mathrm{d}\Omega_{12}$ is the spherical integral of relative angles between $\mathbf{k}_1$ and $\mathbf{k}_2$. The former spherical integral can be directly carried out and results in a factor 4π. To finish the integral with $\mathrm{d}\Omega_{12} = \mathrm{d}\cos\theta_{12}\mathrm{d}\varphi_{12}$, we make use of the remaining delta function, which can be rewritten as

$$\delta(E_q - E_1 - E_2 - E_3) = \frac{E_3}{k_1 k_2} \delta\Big(\cos\theta_{12} - \frac{E_3^2 - k_1^2 - k_2^2 - \mu^2}{2k_1 k_2}\Big), \tag{7.46}$$

by means of $E_3 = \sqrt{k_1^2 + k_2^2 + 2k_1 k_2 \cos\theta + \mu^2}$. Thus

$$\int \mathrm{d}\Omega_1 \mathrm{d}\Omega_{12}\, \delta(E_q - E_1 - E_2 - E_3) = \frac{8\pi^2 E_3}{k_1 k_2}.$$

Now using $k_1 \mathrm{d}k_1 = E_1 \mathrm{d}E_1$ and $k_2 \mathrm{d}k_2 = E_2 \mathrm{d}E_2$, we have

$$\int \mathrm{d}\Pi_3 = \int \frac{\mathrm{d}k_1 \mathrm{d}k_2\, k_1^2 k_2^2}{8(2\pi)^5 E_1 E_2 E_3} \frac{8\pi^2 E_3}{k_1 k_2} = \frac{1}{32\pi^3} \int \mathrm{d}E_1 \mathrm{d}E_2 = \frac{s}{128\pi^3} \int \mathrm{d}x_1 \mathrm{d}x_2. \tag{7.47}$$

To determine the integral region for $m_1 = m_2 = 0$ and $m_3 = \mu$, we note that there are two extremal cases: $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel or antiparallel. In the former case, we have

$$E_q = E_1 + E_2 + E_3 = E_1 + E_2 + \sqrt{(E_1 + E_2)^2 + \mu^2}, \tag{7.48}$$

which yields

$$2E_q(E_1 + E_2) = E_q^2 - \mu^2, \tag{7.49}$$

while in the latter case,

$$E_q = E_1 + E_2 + \sqrt{(E_1 - E_2)^2 + \mu^2}, \tag{7.50}$$

which gives

$$(E_q - 2E_1)(E_q - 2E_2) = \mu^2. \tag{7.51}$$

These two boundary cases can be represented by x_i variables as

$$x_1 + x_2 = 1 - \frac{\mu^2}{s}; \tag{7.52}$$

$$(1 - x_1)(1 - x_2) = \frac{\mu^2}{s}. \tag{7.53}$$

The integral thus goes over the region bounded by these two curves.

(c) Now we calculate the differential cross section for the process $e^+e^- \to q\bar{g}g$ to lowest order in α and α_g. First, the amplitude is

$$\mathrm{i}\mathcal{M} = Q_f(-\mathrm{i}e)^2(-\mathrm{i}g)\epsilon_\nu^*(k_3)\bar{u}(k_1)\left[\gamma^\nu \frac{\mathrm{i}}{\not{k}_1 + \not{k}_3}\gamma^\mu - \gamma^\mu \frac{\mathrm{i}}{\not{k}_2 + \not{k}_3}\gamma^\nu\right] v(k_2) \frac{-\mathrm{i}}{q^2} \bar{v}(p_2)\gamma_\mu u(p_1). \tag{7.54}$$

Then, the squared amplitude is

$$\begin{aligned}
\frac{1}{4}\sum |\mathrm{i}\mathcal{M}|^2 &= \frac{Q_f^2 g^2 e^4}{4s^2}(-g_{\nu\sigma})\,\mathrm{tr}\,(\gamma_\mu \not{p}_1 \gamma_\rho \not{p}_2) \\
&\quad \times \mathrm{tr}\left[\left(\gamma^\nu \frac{1}{\not{k}_1 + \not{k}_3}\gamma^\mu - \gamma^\mu \frac{1}{\not{k}_2 + \not{k}_3}\gamma^\nu\right)\not{k}_2 \left(\gamma^\rho \frac{1}{\not{k}_1 + \not{k}_3}\gamma^\sigma - \gamma^\sigma \frac{1}{\not{k}_2 + \not{k}_3}\gamma^\rho\right)\not{k}_1\right] \\
&= \frac{4Q_f^2 g^2 e^4}{3s^2}(8p_1 \cdot p_2)\left[\frac{4(k_1 \cdot k_2)(k_1 \cdot k_2 + q \cdot k_3)}{(k_1 + k_3)^2 (k_2 + k_3)^2}\right.
\end{aligned}$$

$$+\Big(\frac{1}{(k_1+k_3)^4}+\frac{1}{(k_2+k_3)^4}\Big)\big(2(k_1\cdot k_3)(k_2\cdot k_3)-\mu^2(k_1\cdot k_2)\big)\bigg]. \tag{7.55}$$

We have used the trick described in Peskin's book (P261) when getting through the last equal sign. Now rewrite the quantities of final-state kinematics in terms of x_i, and set $\mu\to 0$, we obtain

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 &= \frac{2Q_f^2g^2e^4}{3s^2}\big(8p_1\cdot p_2\big)\bigg[\frac{2(1-x_3)}{(1-x_1)(1-x_2)}+\frac{1-x_1}{1-x_2}+\frac{1-x_2}{1-x_1}\bigg]\\ &= \frac{8Q_f^2g^2e^4}{3s}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}.\end{aligned} \tag{7.56}$$

Thus the differential cross section, with 3 colors counted, reads

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}x_1\mathrm{d}x_2}\bigg|_{\text{COM}} &= \frac{1}{2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|}\frac{s}{128\pi^3}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)},\end{aligned} \tag{7.57}$$

where we have used the fact that the initial electron and positron are massless, which implies that $2E_{\mathbf{p}_1}=2E_{\mathbf{p}_2}=\sqrt{s}$ and $|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|=2$ in COM frame.

(d) Now we reevaluate the averaged squared amplitude, with μ kept nonzero in (7.55). The result is

$$\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2=\frac{8Q_f^2g^2e^4}{3s}F(x_1,x_2,\mu^2/s), \tag{7.58}$$

where

$$\begin{aligned}F\big(x_1,x_2,\tfrac{\mu^2}{s}\big) &= \frac{2(x_1+x_2-1+\frac{\mu^2}{s})(1+\frac{\mu^2}{s})}{(1-x_1)(1-x_2)}\\ &\quad+\bigg[\frac{1}{(1-x_1)^2}+\frac{1}{(1-x_2)^2}\bigg]\Big((1-x_1)(1-x_2)-\tfrac{\mu^2}{s}\Big).\end{aligned} \tag{7.59}$$

The cross section, then, can be got by integrating over $\mathrm{d}x_1\mathrm{d}x_2$:

$$\begin{aligned}\sigma(e^+e^-\to q\bar{q}g) &= \frac{1}{2E_{\mathbf{p}_1}2E_{\mathbf{p}_2}|v_{\mathbf{p}_1}-v_{\mathbf{p}_2}|}\frac{s}{128\pi^3}\int\mathrm{d}x_1\mathrm{d}x_2\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\int_0^{1-\frac{\mu^2}{s}}\mathrm{d}x_1\int_{1-x_1-\frac{\mu^2}{s}}^{1-\frac{t}{s(1-x_1)}}\mathrm{d}x_2\,F\big(x_1,x_2,\tfrac{\mu^2}{s}\big)\\ &= \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\bigg[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+5-\tfrac{1}{3}\pi^2+\mathcal{O}(\mu^2)\bigg].\end{aligned} \tag{7.60}$$

(e) It is straightforward to finish the integration over Feynman parameters in (a), yielding

$$F_1(q^2=s)=Q_f^2-\frac{Q_f^2\alpha_g}{4\pi}\bigg[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+\tfrac{7}{2}-\tfrac{1}{3}\pi^2-\mathrm{i}\pi\Big(2\log\frac{\mu^2}{s}+7\Big)+\mathcal{O}(\mu^2)\bigg]. \tag{7.61}$$

Then the cross section, to the order of α_g, is given by

$$\sigma(e^+e^-\to q\bar{q})=\frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\bigg\{1-\frac{\alpha_g}{2\pi}\bigg[\log^2\frac{\mu^2}{s}+3\log\frac{\mu^2}{s}+\tfrac{7}{2}-\tfrac{1}{3}\pi^2\bigg]+\mathcal{O}(\mu^2)\bigg\}. \tag{7.62}$$

(f) Combining the results in (d) and (e), we reach the final result:

$$\sigma(e^+e^- \to q\bar{q} + q\bar{q}g) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \left[1 + \frac{3\alpha_g}{4\pi}\right]. \tag{7.63}$$

It is worth noting that all divergent terms as $\mu \to 0$ cancel out in this expression.

Chapter 9

Functional Methods

9.1 Scalar QED

The Lagrangian for scalar QED reads,

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) - m^2\phi^\dagger\phi, \tag{9.1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \mathrm{D}_\mu\phi = (\partial_\mu + \mathrm{i}eA_\mu)\phi. \tag{9.2}$$

(a) Expanding the covariant derivative, it's easy to find the corresponding Feynman's rules:

$$\begin{aligned}(A_\mu\text{-}A_\nu\text{-}\phi^\dagger\text{-}\phi \text{ interaction}) &= 2ie^2\eta^{\mu\nu},\\ (A_\mu\text{-}\phi^\dagger(p_1)\text{-}\phi(p_2) \text{ interaction}) &= -ie(p_1-p_2)^\mu,\end{aligned}$$

with all momenta pointing inwards.

The propagators are standard. We will work in the Feynman gauge and set $\xi = 1$, then the propagator for photon is simply

$$\frac{-i\eta_{\mu\nu}}{p^2+i\epsilon},$$

and the propagator for scalar is

$$\frac{i}{p^2-m^2+i\epsilon}.$$

(b) Now we calculate the spin-averaged differential cross section for the process $e^+e^- \to \phi^*\phi$. The scattering amplitude is given by

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}e)^2\bar{v}(k_2)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{s}(p_1-p_2)_\mu. \tag{9.3}$$

Then the spin-averaged and squared amplitude is

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4s^2}\,\mathrm{tr}\left[(\not{p}_1-\not{p}_2)\not{k}_1(\not{p}_1-\not{p}_2)\not{k}_2\right]$$

$$= \frac{e^4}{4s^2}\big[8(k_1\cdot p_1 - k_1\cdot p_2)(k_2\cdot p_1 - k_2\cdot p_2) - 4(k_1\cdot k_2)(p_1-p_2)^2\big]. \tag{9.4}$$

We may parameterize the momenta as

$$k_1 = (E,0,0,E), \qquad p_1 = (E, p\sin\theta, 0, p\cos\theta),$$
$$k_2 = (E,0,0,-E), \qquad p_2 = (E, -p\sin\theta, 0, -p\cos\theta),$$

with $p = \sqrt{E^2-m^2}$. Then we have

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4p^2}{2E^2}\sin^2\theta. \tag{9.5}$$

Thus the differential cross section is:

$$\Big(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big)_{\text{CM}} = \frac{1}{2(2E)^2}\frac{p}{8(2\pi)^2E}\Big(\tfrac{1}{4}\sum|\mathcal{M}|^2\Big) = \frac{\alpha^2}{8s}\Big(1-\frac{m^2}{E^2}\Big)^{3/2}\sin^2\theta. \tag{9.6}$$

(c)

$$\begin{aligned}
\delta\Pi_{\mu\nu} &= 2\mathrm{i}e^2\eta_{\mu\nu}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2-m^2} - (-\mathrm{i}e)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(p-2k)_\mu(p-2k)_\nu}{(k^2-m^2)((p-k)^2-m^2)}\\
&= -e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{2\eta_{\mu\nu}\big((p-k)^2-m^2\big)-(p-2k)_\mu(p-2k)_\nu}{(k^2-m^2)\big((p-k)^2-m^2\big)}\\
&= -e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{2\eta_{\mu\nu}\big(k'^2+(1-x)^2p^2-m^2\big)+(1-2x)^2p_\mu p_\nu+4k'^\mu k'^\nu}{(k'^2-\Delta)^2}\\
.&= -e^2\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{2\eta_{\mu\nu}k'^2(1-\frac{2}{d})+2\eta_{\mu\nu}\big((1-x)^2p^2-m^2\big)-(1-2x)^2p_\mu p_\nu}{(k'^2-\Delta)^2}\\
&= \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\Big[\frac{(1-\frac{d}{2})\Gamma(1-\frac{d}{2})2\eta_{\mu\nu}}{\Delta^{2-d/2}}\\
&\qquad+\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big(2\eta_{\mu\nu}\big((1-x)^2p^2-m^2\big)-(1-2x)^2p_\mu p_\nu\big)\Big]\\
&= \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\big[2\big((1-x)^2-x(1-x)\big)p^2\eta_{\mu\nu}-(1-2x)^2p_\mu p_\nu\big].
\end{aligned} \tag{9.7}$$

We can symmetrize the integrand as $(1-x)^2 \to \frac{1}{2}\big((1-x)^2+x^2\big)$, then we get

$$\delta\Pi_{\mu\nu} = \frac{-\mathrm{i}e^2}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(1-2x)^2\big(p^2\eta_{\mu\nu}-p_\mu p_\nu\big). \tag{9.8}$$

9.2 Statistical field theory

In this problem we study the path integral formulation in statistical mechanics. The theory can be described by the partition function:

$$Z = \operatorname{tr} e^{-\beta H}, \tag{9.9}$$

where H is the Hamiltonian of the system. It is a function of the generalized coordinates q and the corresponding conjugate momentum p. In this problem, we simply assume the Hamiltonian has the following form:

$$H = \frac{p^2}{2m} + V(q). \tag{9.10}$$

We assume the dimension of the configuration space is d, then both q and p have d components. Then we assume the eigenstates of both q and p form a complete orthonormal basis of the Hilbert space:

$$1 = \int \mathrm{d}^d q\, |q\rangle\langle q|; \qquad 1 = \int \frac{\mathrm{d}^d p}{(2\pi)^d} |p\rangle\langle p|. \tag{9.11}$$

Then the partition function can be written as

$$Z = \operatorname{tr} e^{-\beta H} = \int \mathrm{d}^d q\, \langle q|e^{-\beta H}|q\rangle. \tag{9.12}$$

(a) Now we derive a path integral expression for the partition function. Following the same way of deriving path integral in a quantum field theory, we separate the quantity $e^{-\beta H}$ into N factors:

$$e^{-\beta H} = e^{-\epsilon H} \cdots e^{-\epsilon H}, \qquad \text{(N factors)},$$

then inserting a complete basis between each pair of adjacent factors, as

$$e^{-\beta H} = \int \mathrm{d}^d q_1 \cdots \mathrm{d}^d q_{N-1} \langle q|e^{-\epsilon H}|q_{N-1}\rangle\langle q_{N-1}|e^{-\epsilon H}|q_{N-2}\rangle \cdots \langle q_1|e^{-\epsilon H}|q\rangle.$$

Now we focus on one factors:

$$\langle q_{i+1}|e^{-\epsilon H}|q_i\rangle = \langle q_{i+1}|e^{-\epsilon\left(\frac{1}{2m}p^2 + V(q)\right)}|q_i\rangle = e^{-\epsilon V(q_i)}\langle q_{i+1}|e^{-\frac{\epsilon}{2m}p^2}|q_i\rangle,$$

and

$$\begin{aligned}\langle q_{i+1}|e^{-\frac{\epsilon}{2m}p^2}|q_i\rangle &= \int \frac{\mathrm{d}^d p_{i+1}\mathrm{d}^d p_i}{(2\pi)^d(2\pi)^d}\langle q_{i+1}|p_{i+1}\rangle\langle p_{i+1}|e^{-\epsilon p^2/2m}|p_i\rangle\langle p_i|q_i\rangle \\ &= \int \frac{\mathrm{d}^d p}{(2\pi)^d} e^{ip(q_{i+1}-q_i)} e^{-\epsilon p^2/2m} = \left[\tfrac{m}{2\pi\epsilon}\right]^{d/2} e^{-m(q_{i+1}-q_i)^2/2\epsilon}.\end{aligned}$$

Inserting all this into the partition function, we get:

$$Z = \left[\frac{m}{2\pi\epsilon}\right]^{Nd/2} \prod_{i=0}^{N} \int \mathrm{d}^d q_i \exp\left[-\frac{m(q_{i+1}-q_i)^2}{2\epsilon} - \epsilon V(q_i)\right], \tag{9.13}$$

with $q_{N+1} = q_0$.

Now let $N \to \infty$, then we have

$$Z = \int \mathcal{D}q \exp\left[-\beta \oint \mathrm{d}\tau L_E(\tau)\right], \tag{9.14}$$

where the integral measure is defined by

$$\mathcal{D}q = \lim_{N\to\infty}\Big[\frac{m}{2\pi\epsilon(N)}\Big]^{Nd/2}\prod_{i=0}^{N}\mathrm{d}^d q_i, \tag{9.15}$$

and $L_E(\tau)$ is a Lagrangian in Euclidean form:

$$L_E(\tau) = \frac{m}{2}\Big(\frac{\mathrm{d}q}{\mathrm{d}\tau}\Big)^2 + V(q(\tau)). \tag{9.16}$$

Note that the periodic integral on τ comes from the trace in the partition function.

(b) Now we study an explicit example, a simple harmonic oscillator, which can by defined by the Lagrangian

$$L_E = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\omega^2 q^2. \tag{9.17}$$

Our task is to complete the path integral to find a expression for the partition function of harmonic oscillator. This can be easily done by a Fourier transformation of the coordinates $q(\tau)$ with respect to τ. Since the "time" direction is periodic, the Fourier spectrum of q is discrete. That is,

$$q(\tau) = \beta^{-1/2}\sum_n e^{2\pi\mathrm{i}n\tau/\beta} q_n, \tag{9.18}$$

Then we have:

$$\begin{aligned}\int\mathrm{d}\tau\, L_E(\tau) &= \int\mathrm{d}\tau\frac{1}{2\beta}\sum_{m,n}\Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2 mn + \omega^2\Big] q_m q_n e^{2\pi\mathrm{i}(m+n)\tau/\beta}\\ &= \frac{1}{2}\sum_{m,n}\Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2 mn + \omega^2\Big] q_m q_n \delta_{m,-n} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big[\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega^2\Big] q_n q_{-n}\\ &= \frac{1}{2}\sum_{n\in\mathbb{Z}}\Big[\Big(\frac{2\pi}{\beta}\Big)^2 n^2 + \omega^2\Big]|q_n|^2. \end{aligned}\tag{9.19}$$

Then the path integral can be written as,

$$\begin{aligned} Z &= C\int\mathrm{d}q_0\, e^{-\beta\omega^2 q_0^2}\int\prod_{n>0}\mathrm{dRe}q_n\mathrm{dIm}q_n\,\exp\Big[-\frac{\beta}{2}\Big(\frac{4\pi^2 n^2}{\beta^2}+\omega^2\Big)|q_n|^2\Big]\\ &= \frac{C}{\omega}\prod_{n>0}\Big[\frac{4\pi^2 n^2}{\beta^2}+\omega^2\Big]^{-1} = \frac{C}{\omega}\prod_{n>0}\Big[1+\frac{1}{(\pi n)^2}\Big(\frac{\beta\omega}{2}\Big)^2\Big]^{-1}\\ &= C\sinh^{-1}(\beta\omega/2) = C\sum_{n\geq 0}\exp\big[-\beta\omega(n+\tfrac{1}{2})\big]. \end{aligned}\tag{9.20}$$

(c) From now on we will consider the statistics of fields. We study the statistical properties of boson system, fermion system, and photon system.

For a scalar field, the Lagrangian is given by,

$$L_E(\tau) = \int\mathrm{d}^3x\,\frac{1}{2}\Big[\dot{\phi}^2(\tau,\mathbf{x}) + \big(\nabla\phi(\tau,\mathbf{x})\big)^2 + m^2\phi^2(\tau,\mathbf{x})\Big]. \tag{9.21}$$

Following the method we used to deal with the simple harmonic oscillator, here we decompose the scalar field $\phi(\tau,\mathbf{x})$ into eigenmodes in momentum space:

$$\phi(\tau,\mathbf{x})=\beta^{-1/2}\sum_n e^{2\pi\mathrm{i}n\tau/\beta}\int\frac{\mathrm{d}^3k}{(2\pi)^3}e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\phi_{n,\mathbf{k}}. \tag{9.22}$$

Then the Lagrangian can also be rewritten in terms of modes, as,

$$\begin{aligned}\int\mathrm{d}\tau\,L_E(\tau)=&\int\mathrm{d}\tau\mathrm{d}^3x\sum_{n,n'}\int\frac{\mathrm{d}^3k\mathrm{d}^3k'}{(2\pi)^6}\frac{1}{2\beta}\Big[\Big(\frac{2\pi\mathrm{i}}{\beta}\Big)^2n'n-\mathbf{k}'\cdot\mathbf{k}+m^2\Big]\\&\times\phi_{n,\mathbf{k}}\phi_{n',\mathbf{k}'}e^{\mathrm{i}2\pi(n'+n)\tau/\beta+\mathrm{i}(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}\\=&\,\frac12\sum_n\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Big[\Big(\frac{2\pi}{\beta}\Big)^2n^2+\mathbf{k}^2+m^2\Big]|\phi_{n,\mathbf{k}}|^2\\=&\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Big[\frac12\omega_{\mathbf{k}}^2|\phi_{0,\mathbf{k}}|^2+\sum_{n>0}\Big(\Big(\frac{2\pi}{\beta}\Big)^2n^2+\omega_{\mathbf{k}}^2\Big)|\phi_{n,\mathbf{k}}|^2\Big],\end{aligned} \tag{9.23}$$

where $\omega_{\mathbf{k}}^2=\mathbf{k}^2+m^2$. Then the partition function, as a path integral over the field configurations can be represented by

$$Z=C\int\prod_{n>0,\mathbf{k}}\mathrm{Re}\phi_{n,\mathbf{k}}\mathrm{Im}\phi_{n,\mathbf{k}}\,\exp\Big[-\beta\Big(\Big(\frac{2\pi}{\beta}\Big)^2n^2+\omega_{\mathbf{k}}^2\Big)|\phi_{n,\mathbf{k}}|^2\Big]. \tag{9.24}$$

By the calculation similar to that in (b), we get

$$Z=C\prod_{\mathbf{k}}\Big[\omega_{\mathbf{k}}\prod_{n>0}\Big(\frac{4\pi^2n^2}{\beta^2}+\omega_{\mathbf{k}}^2\Big)\Big]^{-1}=C\prod_{\mathbf{k}}\exp\Big[-\beta\omega_{\mathbf{k}}\big(n+\tfrac12\big)\Big]. \tag{9.25}$$

This product gives the meaning to the formal expression $\big[\det(-\partial^2+m^2)\big]^{-1/2}$ with proper regularization.

(d) Then consider the fermionic oscillator. The action is given by,

$$S=\int\mathrm{d}\tau\,L_E(\tau)=\int\mathrm{d}\tau\,\Big(\bar\psi(\tau)\dot\psi(\tau)+\omega\bar\psi(\tau)\psi(\tau)\Big). \tag{9.26}$$

The antiperiodic boundary condition $\psi(\tau+\beta)=-\psi(\tau)$ is crucial to expanding the fermion into modes:

$$\psi(\tau)=\beta^{-1/2}\sum_{n\in\mathbb{Z}+1/2}e^{2\pi\mathrm{i}\tau/\beta}\psi_n. \tag{9.27}$$

Then the partition function can be evaluated to be

$$\begin{aligned}Z=&\int\prod_n\mathrm{d}\bar\psi_n\mathrm{d}\psi_n\Big[-\beta\sum_{n\in\mathbb{Z}+1/2}\bar\psi_n\Big(\frac{2\pi\mathrm{i}n}{\beta}+\omega\Big)\psi_n\Big]\\=&\,C(\beta)\prod_{n\in\mathbb{Z}+1/2}\Big(\frac{2\pi\mathrm{i}n}{\beta}+\omega\Big)=C(\beta)\prod_{n=0}^{\infty}\Big(\frac{4\pi^2(n+\frac12)^2}{\beta^2}+\omega^2\Big)\\=&\,C(\beta)\cosh\big(\tfrac12\beta\omega\big)=C(\beta)\Big(e^{\beta\omega/2}+e^{-\beta\omega/2}\Big),\end{aligned} \tag{9.28}$$

with the form of a two-level system, as expected.

(e) Finally we consider the system of photons. The partition function is given by

$$
\begin{aligned}
Z &= \int \mathcal{D}A_\mu \mathcal{D}b \mathcal{D}c \, \exp\Big[\int \mathrm{d}\tau \mathrm{d}^3 x \, \big(-\tfrac{1}{2} A_\mu \partial^2 A^\mu - b\partial^2 c \big) \Big] \\
&= C(\beta) \big[\det(-\partial) \big]^{4\cdot(-1/2)} \cdot \det(-\partial^2),
\end{aligned} \tag{9.29}
$$

where the first determinant comes from the integral over the vector field A_μ while the second one comes from the integral over the ghost fields. Therefore,

$$
Z = C(\beta) \big[\det(-\partial^2) \big]^{2\cdot(-1/2)}, \tag{9.30}
$$

which shows the contributions from the two physical polarizations of a photon. Here we see the effect of the ghost fields of eliminating the additional two unphysical polarizations of a vector field.

Chapter 10

Systematics of Renormalization

10.1 One-Loop structure of QED

(a) In this problem we show that any photon n-point amplitude with n an odd number vanishes.

Now we evaluate explicitly the one-point photon amplitude and three-point photon amplitude at 1-loop level to check Furry's theorem. The one-point amplitude at 1-loop level is simply given by,

$$\mathrm{i}\Gamma^{(1)} = (-\mathrm{i}e)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}\,\mathrm{tr}\,[\gamma^\mu(\not{k}+m)]}{k^2-m^2} = 0, \tag{10.1}$$

and the three-point amplitude consists of two diagrams,

$$\begin{aligned}\mathrm{i}\Gamma^{(3)} = (-\mathrm{i}e)^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}(-1)\bigg\{&\mathrm{tr}\Big[\gamma^\mu\frac{\mathrm{i}}{\not{k}-m}\gamma^\nu\frac{\mathrm{i}}{\not{k}+\not{p}_1-m}\gamma^\lambda\frac{\mathrm{i}}{\not{k}+\not{p}_1+\not{p}_2-m}\Big]\\ &+\ \mathrm{tr}\Big[\gamma^\mu\frac{\mathrm{i}}{\not{k}+\not{p}_1+\not{p}_2-m}\gamma^\lambda\frac{\mathrm{i}}{\not{k}+\not{p}_1-m}\gamma^\nu\frac{\mathrm{i}}{\not{k}-m}\Big]\bigg\}.\end{aligned} \tag{10.2}$$

(b) Next we will show that the potential logarithmic divergences in photon four-point diagrams cancel with each other. Since the divergence in this case does not depend on external momenta, we will set all external momenta to be zero for simplicity. For the same reason we will also set the fermion's mass to be zero. Then the six diagrams contributing the four-point amplitude can be evaluated as,

$$\begin{aligned}&(\text{Divergent part of } \mathrm{i}\Gamma^{\mu\nu\rho\sigma})\\ =&\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-1}{(k^2)^4}\Big[\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}]+\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\sigma\not{k}\gamma^\rho\not{k}]+\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\rho\not{k}\gamma^\nu\not{k}\gamma^\sigma\not{k}]\\ &+\ \mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}\gamma^\nu\not{k}]+\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\sigma\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}]+\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\sigma\not{k}\gamma^\rho\not{k}\gamma^\nu\not{k}]\Big].\end{aligned} \tag{10.3}$$

Now let's focus on the first trace, which can be worked out explicitly, to be

$$\mathrm{tr}\,[\gamma^\mu\not{k}\gamma^\nu\not{k}\gamma^\rho\not{k}\gamma^\sigma\not{k}] = 32k^\mu k^\nu k^\rho k^\sigma - 8k^2\big(k^\mu k^\nu g^{\rho\sigma}+k^\rho k^\sigma g^{\mu\nu}+k^\mu k^\sigma g^{\nu\rho}+k^\nu k^\rho g^{\mu\sigma}\big)$$

$$+4(k^2)^2(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}). \tag{10.4}$$

Then, we symmetrize the momentum factors according to $k^\mu k^\nu \to k^2 g^{\mu\nu}/4$ and $k^\mu k^\nu k^\rho k^\sigma \to (k^2)^2(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})/24$. (Since the divergence can be at most logarithmic, so it is safe to set spacetime dimension $d=4$ at this stage.) Then the first trace term reduces to,

$$\text{tr}\,[\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}] \Rightarrow \frac{4}{3}(k^2)^2(g^{\mu\nu}g^{\rho\sigma}-2g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}). \tag{10.5}$$

The other five terms can be easily got by permuting indices. Then it is straightforward to see that the six terms sum to zero.

10.2 Renormalization of Yukawa theory

In this problem we study the pseudoscalar Yukawa Lagrangian,

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}m^2\phi^2+\bar{\psi}(i\not{\partial}-M)\psi-ig\bar{\psi}\gamma^5\psi\phi\,, \tag{10.6}$$

where ϕ is a real scalar and ψ is a Dirac Fermion.

(a) Let's figure out how the superficial degree of divergence D depends on the number of external lines. From power counting, it's easy to see that D can be represented by

$$D=4L-P_f-2P_s\,, \tag{10.7}$$

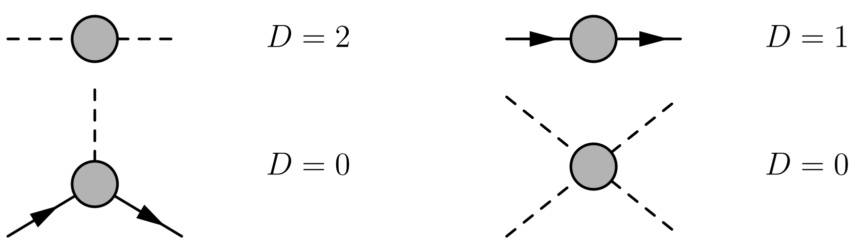
where L is the no. of loops, P_f is the no. of internal fermion lines, and P_s is the no. of internal scalar lines. We also note the following simple relations:

$$\begin{aligned} L &= P_f+P_s-V+1\,,\\ 2V &= 2P_f+N_f\,,\\ V &= 2P_s+N_s\,. \end{aligned}$$

Then we can deduce

$$D=4L-P_f-2P_s=4(P_f+P_s-V+1)-P_f-2P_s=4-\tfrac{3}{2}N_f-N_s\,. \tag{10.8}$$

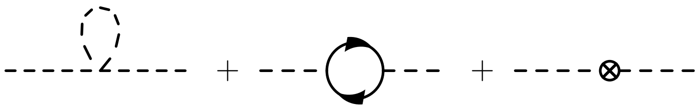
Guided by this result, we can find all divergent amplitudes as follows.



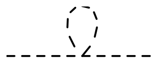
We note that we have ignored the vacuum diagram, which simply contributes an infinitely large constant, the potentially divergent diagrams with odd number of external scalars are also ignored, since they actually vanish. This result shows that the original theory cannot be renormalized unless we including a new ϕ^4 interaction, as

$$\delta\mathcal{L}=-\tfrac{\lambda}{4!}\phi^4\,. \tag{10.9}$$

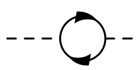
(b) Now let us evaluate the divergent parts of all 1-loop diagrams of Yukawa theory. First we consider the two point function of scalar. The one-loop contribution to this amplitude is shown as follows.



The $d = 4$ pole of first two loop diagrams can be determined as



$$= \frac{-\mathrm{i}\lambda}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2-m^2} \sim \frac{\mathrm{i}\lambda m^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{10.10}$$

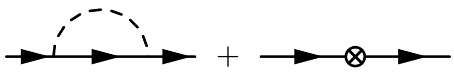


$$= -(-\mathrm{i}g)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\,\mathrm{tr}\left[\frac{\mathrm{i}}{\not{k}-M}\gamma^5\frac{\mathrm{i}}{(\not{k}-\not{p})-M}\gamma^5\right] \sim \frac{4\mathrm{i}g^2(p^2-2M^2)}{(4\pi)^2}\frac{1}{\epsilon}. \tag{10.11}$$

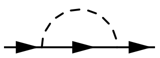
From this we find the divergent part of the counterterm to be

$$\delta_m \sim \frac{(\lambda m^2 - 8g^2M^2)}{(4\pi)^2}\frac{1}{\epsilon}, \qquad \delta_\phi = \frac{-4g^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{10.12}$$

Then we come to the two point function of fermion, the 1-loop correction of which is given by the following two diagrams.



From the pole of the loop diagram

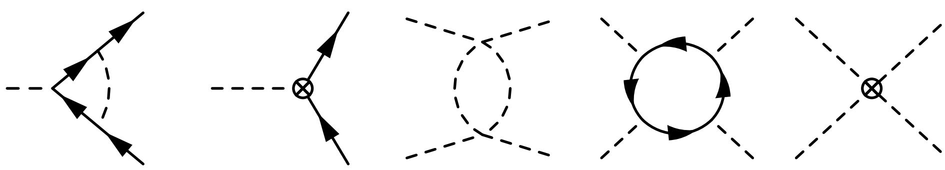


$$= g^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not{k}-M}\gamma^5\frac{\mathrm{i}}{(k-p)^2-m^2} \sim \frac{\mathrm{i}g^2(\not{p}-2M)}{(4\pi)^2}\frac{1}{\epsilon}, \tag{10.13}$$

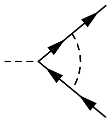
we find the following counterterms:

$$\delta_M \sim \frac{-2g^2M}{(4\pi)^2}\frac{1}{\epsilon}, \qquad \delta_\psi \sim \frac{-g^2}{(4\pi)^2}\frac{1}{\epsilon}. \tag{10.14}$$

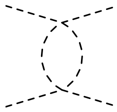
The following two diagrams contribute to 1-loop corrections to Yukawa coupling and ϕ^4 coupling, respectively.



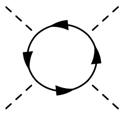
Since the divergent part of diagram is independent of external momenta, we can set all these momenta to be zero. Then the loop diagram is



$$= g^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - M} \gamma^5 \frac{\mathrm{i}}{\not{k} - M} \gamma^5 \frac{\mathrm{i}}{k^2 - m^2} \sim -\frac{g^3\gamma^5}{(4\pi)^2} \frac{2}{\epsilon} \tag{10.15}$$



$$= \frac{(-\mathrm{i}\lambda)^2}{2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \Big(\frac{\mathrm{i}}{k^2 - m^2}\Big)^2 \sim \frac{\mathrm{i}\lambda^2}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.16}$$



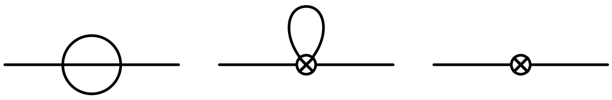
$$= (-1) g^4 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \operatorname{tr} \Big[\Big(\gamma^5 \frac{\mathrm{i}}{\not{k} - M}\Big)^4\Big] \sim -\frac{8\mathrm{i}g^4}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.17}$$

Note that there are 3 permutations for the first diagram and 6 permutations for the second diagram. Then we can determine the divergent part of counterterm to be

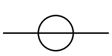
$$\delta_g \sim \frac{2g^3}{(4\pi)^2} \frac{1}{\epsilon}, \qquad \delta_\lambda \sim \frac{3\lambda^2 - 48 g^4}{(4\pi)^2} \frac{1}{\epsilon}. \tag{10.18}$$

10.3 Field-strength renormalization in ϕ^4 theory

In this problem we evaluate the two-loop corrections to scalar's two-point function in ϕ^4 theory in the massless limit. There are three diagrams contribute in total.

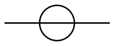


The first diagram reads



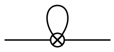
$$\begin{aligned}
&= \frac{(-\mathrm{i}\lambda)^2}{6} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{\mathrm{i}}{k^2 - m^2} \frac{\mathrm{i}}{q^2 - m^2} \frac{\mathrm{i}}{(p-k-q)^2 - m^2} \\
&= \frac{\mathrm{i}\lambda^2}{6} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{\mathrm{d}^d q_E}{(2\pi)^d} \frac{\mathrm{i}}{k_E^2 + m^2} \frac{\mathrm{i}}{q_E^2 + m^2} \frac{\mathrm{i}}{(p_E - k_E - q_E)^2 + m^2} \\
&= \frac{\mathrm{i}\lambda^2}{6} \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{1}{k_E^2 + m^2} \frac{1}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \, \frac{\Gamma(2 - \frac{d}{2})}{[m^2 + x(1-x)(p_E - k_E)^2]^{2-d/2}} \\
&= \frac{\mathrm{i}\lambda^2 \Gamma(2 - \frac{d}{2})}{6(4\pi)^{d/2}} \int_0^1 \mathrm{d}x \, \mathrm{d}y \\
&\quad \times \int \frac{\mathrm{d}^d k_E}{(2\pi)^d} \frac{[x(1-x)]^{d/2-2} (1-y)^{1-d/2} \Gamma(3 - \frac{d}{2}) / \Gamma(2 - \frac{d}{2})}{\big[(k_E - y p_E)^2 + y(1-y) p_E^2 + \big(1 - y + \frac{y}{x(1-x)}\big) m^2\big]^{3-d/2}} \\
&= \frac{\mathrm{i}\lambda^2}{6(4\pi)^d} \int_0^1 \mathrm{d}x \, \mathrm{d}y \, \frac{\Gamma(3-d) [x(1-x)]^{d/2-2} (1-y)^{1-d/2}}{\big[y(1-y) p_E^2 + \big(1 - y + \frac{y}{x(1-x)}\big) m^2\big]^{3-d}}.
\end{aligned} \tag{10.19}$$

Now we take $m^2 = 0$ and $d = 4 - \epsilon \to 4$. Then we have



$$
\begin{aligned}
&= \frac{\mathrm{i}\lambda^2}{12(4\pi)^4}\Gamma(-1+\epsilon)(p_E^2)^{1-\epsilon} + \cdots = -\frac{\mathrm{i}\lambda^2}{12(4\pi)^4}p_E^2\Big[\frac{1}{\epsilon} - \log(p_E^2) + \cdots\Big] \\
&= \frac{\mathrm{i}\lambda^2}{12(4\pi)^4}p^2\Big[\frac{1}{\epsilon} - \log(-p^2) + \cdots\Big]. \qquad (10.20)
\end{aligned}
$$

The second diagram actually vanishes in $m \to 0$ limit. In fact,



$$
= \frac{-\mathrm{i}\delta_\lambda}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2 - m^2} = \frac{-\mathrm{i}\delta_\lambda}{2(4\pi)^{d/2}}\frac{\Gamma(1-\frac{d}{2})}{m^{1-d/2}} \propto m \to 0. \qquad (10.21)
$$

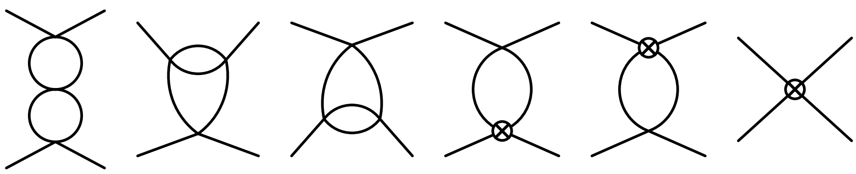
The third diagram reads $\mathrm{i}p^2\delta_Z$. Therefore we can choose the counterterm δ_Z, under the $\overline{MS}$ scheme, to be

$$
\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\Big[\frac{1}{\epsilon} - \log M^2\Big]. \qquad (10.22)
$$

Thus the field strength counterterm receives a nonzero contribution at this order. In the massless limit, it is

$$
\delta_2\Gamma^{(2)} = \frac{\mathrm{i}\lambda^2}{12(4\pi)^4}p^2\log\frac{M^2}{-p^2}. \qquad (10.23)
$$

10.4 Asymptotic behavior of diagrams in ϕ^4 theory



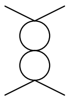
In this problem we calculate the four point amplitude in ϕ^4 theory to 2-loop order in $s \to \infty$, t fixed, limit. The tree level result is simply $-\mathrm{i}\lambda$, and the 1-loop result can be easily evaluated to be

$$
\begin{aligned}
\mathrm{i}\delta_1\mathcal{M} &= \frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2-m^2}\Big[\frac{\mathrm{i}}{(p_s-k)^2-m^2} + \frac{\mathrm{i}}{(p_t-k)^2-m^2} + \frac{\mathrm{i}}{(p_u-k)^2-m^2}\Big] - \mathrm{i}\delta_\lambda \\
&\simeq \frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\Big[3\big(\tfrac{2}{\epsilon} - \gamma + \log 4\pi\big) - \log s - \log t - \log u\Big] - \mathrm{i}\delta_\lambda \\
&= -\frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\big(\log s + \log t + \log u\big) \sim -\frac{\mathrm{i}\lambda^2}{(4\pi)^2}\log s \qquad (10.24)
\end{aligned}
$$

In the last step we take the limit $s \to \infty$. In this limit t can be ignored and $u \simeq -s$. We see the divergent part of the counterterm coefficient δ_λ at 1-loop order is

$$
\delta_\lambda \sim \frac{3\lambda^2}{(4\pi)^2}\frac{1}{\epsilon}. \qquad (10.25)
$$

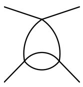
Now we consider the two-loop correction.



$$= \frac{(-\mathrm{i}\lambda)^3}{4}\left[\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}\right]^2 = -\frac{\mathrm{i}\lambda^3}{4(4\pi)^d}\left[\int_0^1 \mathrm{d}x \frac{\Gamma(2-\frac{d}{2})}{[-x(1-x)s]^{2-d/2}}\right]^2$$

$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\log s+\frac{1}{2}\log^2 s\Big). \tag{10.26}$$

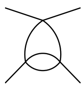
In the last line we only keep the divergent terms as $\epsilon \to 0$ and $s \to \infty$.



$$\begin{aligned}
&= \frac{(-\mathrm{i}\lambda)^3}{2}\int \frac{\mathrm{d}^d k\mathrm{d}^d q}{(2\pi)^{2d}}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}\frac{\mathrm{i}}{q^2}\frac{\mathrm{i}}{(k-p_3-q)^2}\\
&= -\frac{\mathrm{i}\lambda^3}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2(p_s-k)^2}\left[\frac{\mathrm{i}}{(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)(k_E-p_{3E})^2]^{2-d/2}}\right]\\
&= \frac{\mathrm{i}\lambda^3}{2(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)]^{2-d/2}}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\frac{1}{k_E^2(p_{sE}-k_E)^2[(k_E-p_{3E})^2]^{2-d/2}}\\
&= \frac{\mathrm{i}\lambda^3}{2(4\pi)^{d/2}}\int_0^1 \mathrm{d}x\frac{\Gamma(2-\frac{d}{2})}{[x(1-x)]^{2-d/2}}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\int_0^1 \mathrm{d}y\int_0^{1-y}\mathrm{d}z\,\frac{z^{1-d/2}}{(k_E^2+\Delta)^{4-d/2}}\frac{\Gamma(4-\frac{d}{2})}{\Gamma(2-\frac{d}{2})}\\
&= \frac{\mathrm{i}\lambda^3}{2(4\pi)^d}\int \mathrm{d}x\mathrm{d}y\mathrm{d}z\,\frac{z^{1-d/2}}{[x(1-x)]^{2-d/2}}\frac{\Gamma(4-d)}{\Delta^{4-d}},
\end{aligned} \tag{10.27}$$

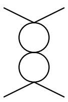
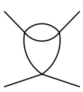
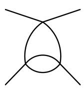
where $\Delta = ys + zp_{3E}^2 - (yp_{sE}+zp_{3E})^2$.

Then we find



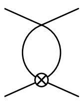
$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\log s+\frac{1}{2}\log^2 s\Big). \tag{10.28}$$

The same result for the third diagram. Then we have

 +  + 

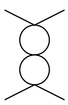
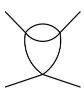
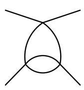
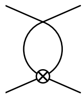
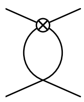
$$\sim -\frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{3}{\epsilon^2}-\frac{3}{\epsilon}\log s+\frac{3}{2}\log^2 s\Big). \tag{10.29}$$

Now we come to the counterterm. The fourth diagram reads



$$\begin{aligned}
&= \frac{(-\mathrm{i}\lambda)(-\mathrm{i}\delta_\lambda)}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_s-k)^2}\\
&\sim \frac{3\lambda^3}{2(4\pi)^2}\frac{1}{\epsilon}\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\Big(1-\frac{\epsilon}{2}\log s+\frac{\epsilon^2}{8}\log^2 s+\cdots\Big)\\
&\sim \frac{3\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{1}{\epsilon^2}-\frac{1}{2\epsilon}\log s+\frac{1}{8}\log^2 s\Big)
\end{aligned} \tag{10.30}$$

The same result for the fifth diagram. Then we have

 +  +  +  + 

$$\sim \frac{\mathrm{i}\lambda^3}{(4\pi)^4}\Big(\frac{3}{\epsilon^2}-\frac{3}{4}\log^2 s\Big). \tag{10.31}$$

So much for the s-channel. The t and u-channel results can be obtained by replacing s with t and u respectively. In the limit $s \to \infty$ and t-fixed, we can simply ignore t and treating $u \sim -s$, then the total 2-loop correction in this limit is

$$\mathrm{i}\delta_2\mathcal{M} \sim -\frac{3\mathrm{i}\lambda^3}{2(4\pi)^4}\log^2 s. \tag{10.32}$$

The double pole $1/\epsilon^2$ has been absorbed by δ_λ.

In summary, we have the following asymptotic expression for the 4-point amplitude to 2-loop order in the $s \to \infty$ and t-fixed limit:

$$\mathrm{i}\mathcal{M} = -\mathrm{i}\lambda - \frac{\mathrm{i}\lambda^2}{(4\pi)^2}\log s - \frac{3\mathrm{i}\lambda^3}{2(4\pi)^4}\log^2 s + \cdots . \tag{10.33}$$

Chapter 11

Renormalization and Symmetry

11.1 Spin-wave theory

(a) Firstly we prove the following formula:

$$\left\langle Te^{\mathrm{i}\phi(x)}e^{-\mathrm{i}\phi(0)}\right\rangle = e^{[D(x)-D(0)]}. \tag{11.1}$$

Where $D(x) = \langle T\phi(x)\phi(0)\rangle$ is the time-ordered correlation of two scalars. The left hand side of this equation can be represented by path integral, as

$$\frac{1}{Z[0]}\int \mathcal{D}\phi\, e^{\mathrm{i}\phi(x)}e^{-\mathrm{i}\phi(0)} \exp\Big[\mathrm{i}\int \mathrm{d}^d x \mathrm{d}^d y\, \tfrac{1}{2}\phi(x)D^{-1}(x-y)\phi(y)\Big]. \tag{11.2}$$

This expression precisely has the form $Z[J]/Z[0]$, with $J(y) = \delta(y-x) - \delta(0)$. Thus we have

$$Z[J]/Z[0] = -\frac{1}{2}\int \mathrm{d}^d x \mathrm{d}^d y\, J(x)D(x-y)J(y) = \exp\big[D(x)-D(0)\big], \tag{11.3}$$

which is just the right hand side of the formula.

(b) The operator being translational invariant $O[\phi(x)] = O[\phi(x)-\alpha]$ can depend on ϕ only through $\nabla_\mu\phi$. And the only relevant/marginal Lorentz-invariant operator satisfying this condition is $\frac{1}{2}\rho(\nabla\phi)^2$.

(c) From now on we use bold $\mathbf{x}$ to denote coordinate and italic x to denote its length, $x \equiv |\mathbf{x}|$. We can use the result in (a) to evaluate $\langle s(\mathbf{x})s^*(0)\rangle$, as

$$\langle s(\mathbf{x})s^*(0)\rangle = A^2\langle e^{\mathrm{i}\phi(\mathbf{x})}e^{-\mathrm{i}\phi(0)}\rangle = A^2 e^{D(\mathbf{x})-D(0)}. \tag{11.4}$$

Note that the correlation function

$$D(\mathbf{x}) = \frac{1}{\rho}\int \frac{\mathrm{d}^d k_E}{(2\pi)^d}\frac{1}{k_E^2}e^{\mathrm{i}\mathbf{k}_E\cdot\mathbf{x}} \tag{11.5}$$

is the solution to the following equation:

$$-\rho\nabla^2 D(\mathbf{x}-\mathbf{y}) = \delta^{(d)}(\mathbf{x}-\mathbf{y}). \tag{11.6}$$

Since $D(x)$ is a function of the length only, namely $D(\mathbf{x}) = D(x)$, thus we have

$$-\frac{\rho}{x^{d-1}}\frac{\partial}{\partial x}\Big(x^{d-1}\frac{\partial}{\partial x}D(x)\Big) = \frac{\Gamma(1+\frac{d}{2})}{d\pi^{d/2}}\frac{\delta(x)}{x^{d-1}}. \tag{11.7}$$

Then it's easy to find

$$D(x) = \begin{cases} \dfrac{\Gamma(1+\frac{d}{2})}{d(d-2)\pi^{d/2}\rho}\dfrac{1}{x^{d-2}}, & \text{for } \ d\neq 2,\\ -\dfrac{1}{2\pi\rho}\log x, & \text{for } \ d=2. \end{cases} \tag{11.8}$$

Then we have

Dimension d	$d=1$	$d=2$	$d=3$	$d=4$
$D(x)$	$-\dfrac{1}{2\rho}x$	$-\dfrac{1}{2\pi\rho}\log x$	$\dfrac{1}{4\pi\rho x}$	$\dfrac{1}{4\pi^2\rho x^2}$
$\langle ss^*\rangle$	$\sim e^{-x}$	$\sim 1/x^{2\pi\rho}$	$\sim e^{1/x}$	$\sim e^{1/x^2}$

Since $\rho\to 0$ when $d\to 2$, the correlation function $\langle ss^*\rangle$ in this case is independent of length x.

11.2 A zeroth-order natural relation

We study $N=2$ linear sigma model coupled to fermions:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi^i\partial^\mu\phi^i + \tfrac{1}{2}\mu^2\phi^i\phi^i - \tfrac{1}{4}\lambda(\phi^i\phi^i)^2 + \bar\psi(\mathrm{i}\partial\!\!\!/)\psi - g\bar\psi(\phi^1+\mathrm{i}\gamma^5\phi^2)\psi, \tag{11.9}$$

with ϕ^i a two-component field, $i=1,2$.

(a) Now, under the following transformation:

$$\phi^1\to\phi^1\cos\alpha-\phi^2\sin\alpha;\qquad \phi^2\to\phi^1\sin\alpha+\phi^2\cos\alpha;\qquad \psi\to e^{-\mathrm{i}\alpha\gamma^2/2}\psi, \tag{11.10}$$

the first three terms involving ϕ^i only keep invariant. The fourth term, as the kinetic term of a chiral fermion, is also unaffected by this transformation. Thus, to show the whole Lagrangian is invariant, we only need to check the last term, and this is really the case:

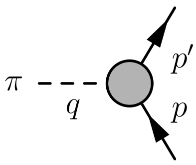
$$\begin{aligned} & -g\bar\psi(\phi^1+\mathrm{i}\gamma^5\phi^2)\psi \\ \to & -g\bar\psi e^{-\mathrm{i}\alpha\gamma^5/2}\big[(\phi^1\cos\alpha-\phi^2\sin\alpha)+\mathrm{i}\gamma^5(\phi^1\sin\alpha+\phi^2\cos\alpha)\big]e^{-\mathrm{i}\alpha\gamma^5/2}\psi \\ = & -g\bar\psi e^{-\mathrm{i}\alpha\gamma^5/2}e^{\mathrm{i}\alpha\gamma^5}(\phi^1+\mathrm{i}\gamma^5\phi^2)e^{-\mathrm{i}\alpha\gamma^5/2}\psi = -g\bar\psi(\phi^1+\mathrm{i}\gamma^5\phi^2)\psi. \end{aligned} \tag{11.11}$$

(b) Now let ϕ acquire a vacuum expectation value v, which equals to $\sqrt{\mu^2/\lambda}$ classically. Then, in terms of new variables $\phi = (v + \sigma(x), \pi(x))$, the Lagrangian reads

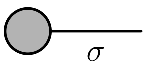
$$\begin{aligned}\mathcal{L} = &\ \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 - \mu^2\sigma^2 - \tfrac{1}{4}\lambda(\sigma^4 + \pi^4) \\ &- \tfrac{1}{2}\lambda\sigma^2\pi^2 - \lambda v\sigma^3 - \lambda v\sigma\pi^2 + \bar{\psi}(\mathrm{i}\not{\partial} - gv)\psi - g\bar{\psi}(\sigma + \mathrm{i}\gamma^5\pi)\psi. \end{aligned} \tag{11.12}$$

That is, the fermion acquire a mass $m_f = gv$.

(c) Now we calculate the radiative corrections to the mass relation $m_f = gv$. The renormalization conditions we need are as follows.

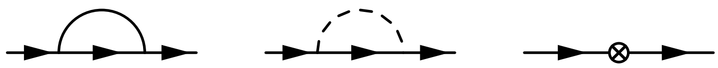


$$= g\gamma^5 \qquad \text{at } q^2 = 0,\ p^2 = p'^2 = m_f^2. \tag{11.13}$$



$$= 0. \tag{11.14}$$

These two conditions fixed g and v so that they receive no radiative corrections. Then we want to show that the mass of the fermion m_f receives *finite* radiative correction at 1-loop. Since the tadpole diagrams of σ sum to zero by the renormalization condition above, the fermion's self-energy receive nonzero contributions from the following three diagrams:



The first two 1-loop diagrams can be evaluated as

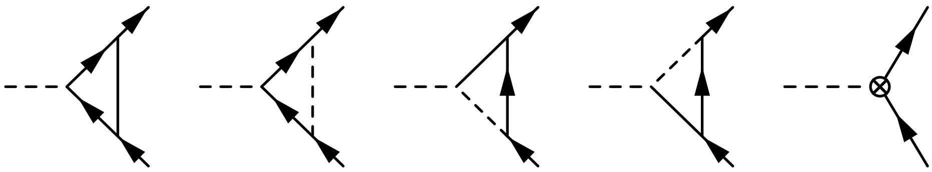
$$\begin{aligned}(\mathrm{e}) &= (-\mathrm{i}g)^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{\not{k} - m_f} \frac{\mathrm{i}}{(k-p)^2 - 2\mu^2} = g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{x\not{p} + m_f}{(k'^2 - \Delta_1)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \frac{\Gamma(2 - \frac{d}{2})}{\Delta_1^{2-d/2}} (x\not{p} + m_f) \\ &= \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x\, (x\not{p} + m_f) \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_1\Big] \end{aligned} \tag{11.15}$$

$$\begin{aligned}(\mathrm{f}) &= g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \gamma^5 \frac{\mathrm{i}}{\not{k} - m_f} \gamma^5 \frac{\mathrm{i}}{(k-p)^2} = g^2 \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}x\, \frac{x\not{p} - m_f}{(k'^2 - \Delta_2)^2} \\ &= \frac{\mathrm{i}g^2}{(4\pi)^{d/2}} \int_0^1 \mathrm{d}x\, \frac{\Gamma(2 - \frac{d}{2})}{\Delta_2^{2-d/2}} (x\not{p} - m_f) \\ &= \frac{\mathrm{i}g^2}{(4\pi)^2} \int_0^1 \mathrm{d}x\, (x\not{p} - m_f) \Big[\tfrac{2}{\epsilon} - \gamma + \log 4\pi - \log \Delta_2\Big]. \end{aligned} \tag{11.16}$$

This leads to

$$(\mathrm{e})+(\mathrm{f})=\frac{\mathrm{i}g^2}{(4\pi)^2}\int_0^1\mathrm{d}x\left\{2x\not{p}\Big[\tfrac{2}{\epsilon}-\gamma+\log 4\pi-\tfrac{1}{2}\log(\Delta_1\Delta_2)\Big]+m_f\log\frac{\Delta_2}{\Delta_1}\right\} \tag{11.17}$$

We see that the correction to the fermions mass m_f from these two diagrams is finite. Besides, the third diagram, namely the counterterm, contributes the mass' correction through $\delta_g v$. The the total correction to m_f is finite only when δ_g is finite. Let us check this by means of the first renormalization condition (11.13) stated above. The 1-loop contributions to (11.13) are as follows.



$$\begin{aligned}(\mathrm{a})=&(-\mathrm{i}g)^2g\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{\not{k}-m_f}\gamma^5\frac{\mathrm{i}}{\not{k}-m_f}\frac{\mathrm{i}}{(k-p)^2-2\mu^2}\\
=&\,\mathrm{i}g^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{(\not{k}+m_f)\gamma^5(\not{k}+m_f)}{(k^2-m_f^2)^2\big((k-p)^2-2\mu^2\big)}=-\mathrm{i}g^3\gamma^5\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2-m_f^2)\big((k-p)^2-2\mu^2\big)}\\
=&-\mathrm{i}g^3\gamma^5\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{1}{(k'^2-\Delta_1)^2}=\frac{g^3\gamma^5}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta_1^{2-d/2}}\\
=&\,\frac{g^3\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\left[\tfrac{2}{\epsilon}-\gamma+\log 4\pi-\log\Delta_1\right]\end{aligned} \tag{11.18}$$

$$\begin{aligned}(\mathrm{b})=&\,g^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not{k}-m_f}\gamma^5\frac{\mathrm{i}}{\not{k}-m_f}\gamma^5\frac{\mathrm{i}}{(k-p)^2-2\mu^2}\\
=&-\mathrm{i}g^3\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\gamma^5(\not{k}+m_f)\gamma^5(\not{k}+m_f)\gamma^5}{(k^2-m_f^2)^2\big((k-p)^2-2\mu^2\big)}=\mathrm{i}g^3\gamma^5\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2-m_f^2)\big((k-p)^2-2\mu^2\big)}\\
=&\,\mathrm{i}g^3\gamma^5\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\,\frac{1}{(k'^2-\Delta_2)^2}=\frac{-g^3\gamma^5}{(4\pi)^{d/2}}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta_2^{2-d/2}}\\
=&\,\frac{-g^3\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\left[\tfrac{2}{\epsilon}-\gamma+\log 4\pi-\log\Delta_2\right]\end{aligned} \tag{11.19}$$

$$\begin{aligned}(\mathrm{c})=&\,(-\mathrm{i}g)g(-2\mathrm{i}\lambda v)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\gamma^5\frac{\mathrm{i}}{\not{k}-m_f}\frac{\mathrm{i}}{(k-p)^2-2\mu^2}\frac{\mathrm{i}}{(k-p)^2}\\
=&\,4\mathrm{i}g^2\lambda v\gamma^5\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(x+y)\not{p}+m_f}{(k'^2-\Delta_3)^3}\\
=&\,\frac{2g^2\lambda v\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(x+y)\not{p}+m_f}{\Delta_3}\end{aligned} \tag{11.20}$$

$$
\begin{aligned}
(\mathrm{d}) &= (-\mathrm{i}g)g(-2\mathrm{i}\lambda v)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{\not{k}-m_f}\gamma^5\frac{\mathrm{i}}{(k-p)^2-2\mu^2}\frac{\mathrm{i}}{(k-p)^2} \\
&= 4\mathrm{i}g^2\lambda v\gamma^5\int\frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{-(x+y)\not{p}+m_f}{(k'^2-\Delta_3)^3} \\
&= \frac{2g^2\lambda v\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{-(x+y)\not{p}+m_f}{\Delta_3}
\end{aligned}
\tag{11.21}
$$

Thus,

$$
(\mathrm{a})+(\mathrm{b})+(\mathrm{c})+(\mathrm{d}) = \frac{g\gamma^5}{(4\pi)^2}\int_0^1\mathrm{d}x\Big[g^2\log\frac{\Delta_2}{\Delta_1}+4\lambda\int_0^{1-x}\mathrm{d}y\,\frac{m_f^2}{\Delta_3}\Big]. \tag{11.22}
$$

11.3 The Gross-Neveu model

The Gross-Neveu Model is a theory of fermions in $1+1$ dimensional spacetime:

$$
\mathcal{L} = \bar{\psi}_i\mathrm{i}\not{\partial}\psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2, \tag{11.23}
$$

with $i = 1,\cdots,N$. The gamma matrices are taken as $\gamma^0=\sigma^2$, $\gamma^1 = i\sigma^1$, where σ^i is the familiar Pauli matrices. We also define $\gamma^5=\gamma^0\gamma^1=\sigma^3$.

(a) The theory is invariant under the transformation $\psi_i\to\gamma^5\psi_i$. It is straightforward to check this. We note that:

$$
\bar{\psi}_i = \psi_i^\dagger\gamma^0 \to \psi_i^\dagger\gamma^5\gamma^0 = -\bar{\psi}_i\gamma^5, \tag{11.24}
$$

thus:

$$
\begin{aligned}
\mathcal{L}\to &-\bar{\psi}_i\gamma^5\mathrm{i}\not{\partial}\gamma^5\psi_i + \tfrac{1}{2}g^2(-\bar{\psi}_i\gamma^5\gamma^5\psi_i)^2 \\
&= \bar{\psi}_i\mathrm{i}\not{\partial}\psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2.
\end{aligned}
\tag{11.25}
$$

However, a mass term will transform as $m_i\bar{\psi}_i\psi_i\to -m_i\bar{\psi}_i\psi_i$, thus a theory respecting this chiral symmetry does not allow such a mass term.

(b) The superficial renormalizability of the theory (by power counting) is obvious since $[g]=0$.

(c) The model can be phrased in another equivalent way:

$$
Z = \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\sigma\exp\bigg[\mathrm{i}\int\mathrm{d}^2x\,\big(\bar{\psi}_i\mathrm{i}\not{\partial}\psi_i - \tfrac{1}{2g^2}\sigma^2-\sigma\bar{\psi}_i\psi_i\big)\bigg]. \tag{11.26}
$$

This can be justified by integrating out σ,

$$
\int\mathcal{D}\sigma\exp\bigg[\mathrm{i}\int\mathrm{d}^2x\,(-\tfrac{1}{2g^2}\sigma^2-\sigma\bar{\psi}_i\psi_i)\bigg] = N\exp\bigg[\mathrm{i}\int\mathrm{d}^2x\,\tfrac{g^2}{2}(\bar{\psi}_i\psi_i)^2\bigg]. \tag{11.27}
$$

which recovers the following path integral:

$$
Z = \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\bigg[\mathrm{i}\int\mathrm{d}^2x\,\big(\bar{\psi}_i\mathrm{i}\not{\partial}\psi_i+\tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2\big)\bigg]. \tag{11.28}
$$

(d) We can also integrate out the fermions ψ_i to get the effective potential for the auxiliary field σ:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\mathrm{i}\int \mathrm{d}^2x\big(\bar{\psi}_i\mathrm{i}\partial\!\!\!/\psi_i - \sigma\bar{\psi}_i\psi_i\big)\right] = \big[\det(\mathrm{i}\partial\!\!\!/ - \sigma)\big]^N = \big[\det(\partial^2+\sigma^2)\big]^N$$
$$= \exp\left[\int \frac{\mathrm{d}^2k}{(2\pi)^2} N\log(-k^2+\sigma^2)\right]. \tag{11.29}$$

The integral is divergent, which should be regularized. We use the dimensional regularization:

$$\int \frac{\mathrm{d}^dk_E}{(2\pi)^d} N\log(k_E^2+\sigma^2) = N\int \frac{\mathrm{d}^dk_E}{(2\pi)^d}\left[\frac{\partial}{\partial\alpha}\frac{1}{k_E^2+\sigma^2}\right]_{\alpha=0}$$
$$= -\mathrm{i}N\frac{\Gamma(-d/2)(\sigma^2)^{d/2}}{(4\pi)^{d/2}}. \tag{11.30}$$

Now we set $d = 2-\epsilon$ and send $\epsilon \to 0$,

$$\int \frac{\mathrm{d}^dk_E}{(2\pi)^d} N\log(k_E^2+\sigma^2) = \frac{\mathrm{i}N\sigma^2}{4\pi}\bigg(\frac{2}{\epsilon} - \gamma + \log 4\pi - \log\sigma^2 + 1\bigg). \tag{11.31}$$

Thus the effective potential is

$$V_{\text{eff}}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2\bigg(\log\frac{\sigma^2}{\mu^2} - 1\bigg) \tag{11.32}$$

by modified minimal subtraction.

(e) Now we minimize the effective potential:

$$0 = \frac{\partial V_{\text{eff}}}{\partial\sigma} = \frac{1}{g^2}\sigma + \frac{N}{2\pi}\sigma\log\frac{\sigma^2}{\mu^2}, \tag{11.33}$$

and find nonzero vacuum expectation values $\langle\sigma\rangle = \pm\mu e^{-\pi/g^2N}$. The dependence of this result on the renormalization condition is totally in the dependence on the subtraction point μ.

(f) It is well-known that the loop expansion is equivalent to the expansion in powers of $\hbar$ in generic perturbation theory around a classical vacuum. This is true because the integrand of the partition function can be put into the form of $e^{\mathrm{i}S/\hbar}$. That is, $\hbar$ appears as an overall coefficient of the action. In our case, we see that the overall factor N plays the same role. Thus by the same argument, we conclude that the loop expansion is equivalent to the expansion in powers of $1/N$. More details can be found in Section III.3 of [4] and Chapter 8 "$1/N$" of [5].

Chapter 12

The Renormalization Group

12.1 Beta Function in Yukawa Theory

In this problem we calculate the 1-loop beta functions in Yukawa theory. All needed ingredients have been given in Problem 10.2 Here we list the needed counterterms:

$$\delta_\psi = -\frac{g^2}{2(4\pi^2)}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.1}$$

$$\delta_\phi = -\frac{2g^2}{(4\pi^2)}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.2}$$

$$\delta_g = \frac{g^3}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right); \tag{12.3}$$

$$\delta_\lambda = \frac{3\lambda^2 - 48g^4}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \log M^2\right). \tag{12.4}$$

Here Λ is the UV cutoff and M is the renormalization scale. Then, the beta functions to lowest order are given by

$$\beta_g = M\frac{\partial}{\partial M}\Big(- \delta_g + \tfrac{1}{2}g_0\delta_\phi + g_0\delta_\psi\Big) = \frac{5g^3}{(4\pi)^2}; \tag{12.5}$$

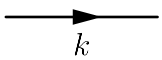
$$\beta_\lambda = M\frac{\partial}{\partial M}\Big(- \delta_\lambda + 2\lambda_0\delta_\phi\Big) = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{(4\pi)^2}. \tag{12.6}$$

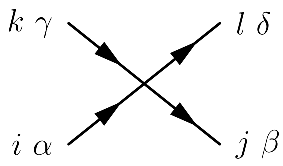
12.2 Beta Function of the Gross-Neveu Model

We evaluate the β function of the 2-dimensional Gross-Neveu model with the Lagrangian

$$\mathcal{L} = \bar{\psi}_i(\mathrm{i}\partial\!\!\!/)\psi_i + \tfrac{1}{2}g^2(\bar{\psi}_i\psi_i)^2, \qquad (i = 1, \cdots, N) \tag{12.7}$$

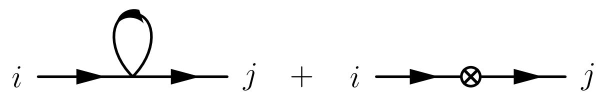
to 1-loop order. The Feynman rules can be easily worked out to be

$i\ \alpha$  $j\ \beta = \left(\dfrac{\mathrm{i}}{k\!\!\!/}\right)_{\beta\alpha}\delta_{ij}$



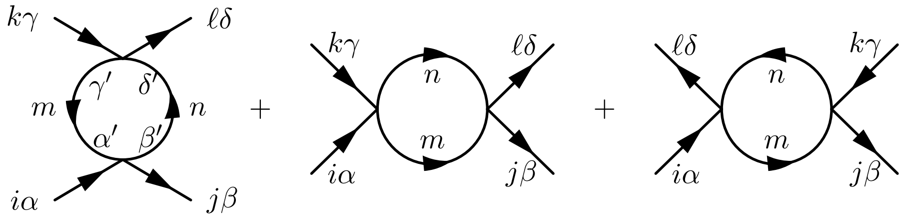
$$= \mathrm{i}g^2(\delta_{ij}\delta_{k\ell}\epsilon_{\beta\alpha}\epsilon_{\delta\gamma} + \delta_{i\ell}\delta_{jk}\epsilon_{\delta\alpha}\epsilon_{\beta\gamma})$$

Now consider the two-point function $\Gamma_{ij}^{(2)}(p)$. The one-loop correction to $\Gamma_{ij}^{(2)}(p)$ comes from the following two diagrams:



It is easy to see the loop diagram contains a factor of $\int \mathrm{d}^2k\ \mathrm{tr}\,[\not{k}^{-1}]$, which is zero under dimensional regularization. Thus the wave function renormalization factor receives no contribution at 1-loop level, namely $\delta_\psi = 0$.

Then we turn to the 4-point function $\Gamma^{(4)}_{ijk\ell}$. There are three diagrams in total, namely,



(a) (b) (c)

We calculate them in turn. The first one:

$$\begin{aligned}(\mathrm{a}) = (\mathrm{i}g^2)^2 &\int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mn}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\gamma'\delta'} + \delta_{n\ell}\delta_{mk}\epsilon_{\delta\delta'}\epsilon_{\gamma'\gamma})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\delta'\beta'} \\ &\times (\delta_{ij}\delta_{mn}\epsilon_{\beta'\alpha'}\epsilon_{\beta\alpha} + \delta_{in}\delta_{jm}\epsilon_{\beta'\alpha}\epsilon_{\beta\alpha'})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\alpha'\gamma'} \\ = g^4\Big(&(-2N+2)\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha} + \tfrac{1}{2}\delta_{i\ell}\delta_{jn}(\gamma^\mu)_{\delta\alpha}(\gamma_\mu)_{\beta\gamma}\Big)\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\end{aligned} \tag{12.8}$$

The second diagram reads:

$$\begin{aligned}(\mathrm{b}) = \frac{1}{2}\cdot(\mathrm{i}g^2)^2 &\int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mj}\delta_{n\ell}\epsilon_{\delta\delta'}\epsilon_{\beta\beta'} + \delta_{m\ell}\delta_{nj}\epsilon_{\delta\beta'}\epsilon_{\beta\delta'})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\beta'\alpha'} \\ &\times (\delta_{im}\delta_{kn}\epsilon_{\alpha'\alpha}\epsilon_{\gamma'\gamma} + \delta_{in}\delta_{km}\epsilon_{\gamma'\alpha}\epsilon_{\alpha'\gamma})\Big(\frac{\mathrm{i}}{-\not{k}}\Big)_{\delta'\gamma'} \\ = -\frac{g^4}{2}&\Big(\delta_{ij}\delta_{k\ell}(\gamma^\mu)_{\delta\gamma}(\gamma_\mu)_{\beta\alpha} + \delta_{i\ell}\delta_{jn}(\gamma^\mu)_{\delta\alpha}(\gamma_\mu)_{\beta\gamma}\Big)\int \frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\end{aligned} \tag{12.9}$$

The third diagram:

$$\begin{aligned}(\mathrm{c}) = (\mathrm{i}g^2)^2 &\int \frac{\mathrm{d}^dk}{(2\pi)^d}(\delta_{mj}\delta_{nk}\epsilon_{\beta\beta'}\epsilon_{\gamma'\gamma} + \delta_{mn}\delta_{jk}\epsilon_{\gamma'\beta'}\epsilon_{\beta\gamma})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\beta'\alpha'} \\ &\times (\delta_{im}\delta_{\ell n}\epsilon_{\delta\delta'}\epsilon_{\alpha'\alpha} + \delta_{i\ell}\delta_{mn}\epsilon_{\alpha'\delta'}\epsilon_{\delta\alpha})\Big(\frac{\mathrm{i}}{\not{k}}\Big)_{\delta'\gamma'}\end{aligned}$$

$$=g^4\Big(\tfrac{1}{2}\delta_{ij}\delta_{k\ell}(\gamma^\mu)_{\delta\gamma}(\gamma_\mu)_{\beta\alpha}+(2-2N)\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha}\Big)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2} \tag{12.10}$$

Summing up the three diagrams and using dimensional regularization with $d=2-\epsilon$, we get

$$\begin{aligned}&-2g^4(N-1)(\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha}+\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha})\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{k^2}\\ &\sim\frac{2(N-1)\mathrm{i}g^4}{4\pi}\frac{2}{\epsilon}(\delta_{ij}\delta_{k\ell}\epsilon_{\delta\gamma}\epsilon_{\beta\alpha}+\delta_{i\ell}\delta_{jk}\epsilon_{\beta\gamma}\epsilon_{\delta\alpha}).\end{aligned} \tag{12.11}$$

Only the divergent terms are kept in the last expression, from which we can read the counterterm

$$\delta_g=-\frac{(N-1)g^4}{2\pi}\Big(\frac{2}{\epsilon}-\log M^2\Big). \tag{12.12}$$

Thus the β function is

$$\beta(g^2)=M\frac{\partial}{\partial M}(-\delta_g)=-\frac{(N-1)(g^2)^2}{\pi}, \tag{12.13}$$

and

$$\beta(g)=-\frac{(N-1)g^3}{2\pi}. \tag{12.14}$$

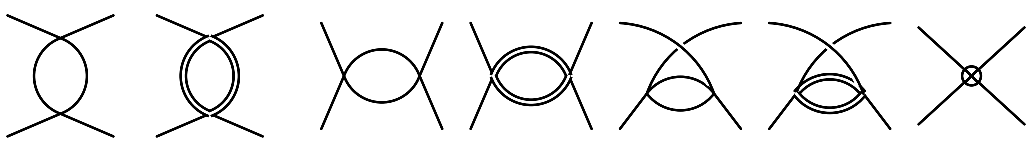
It is interesting to see that the 1-loop β function vanishes for $N=1$. This is because we have the Fierz identity $2(\bar\psi\psi)(\bar\psi\psi)=-(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$, and the Gross-Neveu model in this case is equivalent to massless Thirring model, which is known to have vanishing β function.

12.3 Asymptotic Symmetry

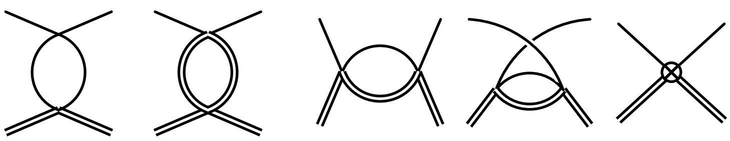
In this problem we study a bi-scalar model, given by the following Lagrangian:

$$\mathcal{L}=\tfrac{1}{2}\big((\partial_\mu\phi_1)^2+(\partial_\mu\phi_2)^2\big)-\tfrac{\lambda}{4!}(\phi_1^4+\phi_2^4)-\tfrac{\rho}{12}\phi_1^2\phi_2^2. \tag{12.15}$$

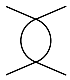
(a) First, we calculate the 1-loop beta functions β_λ and β_ρ. The relevant 1-loop diagrams for calculating β_λ are:



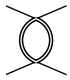
The relevant diagrams for calculating β_ρ are:



Here the single line represents ϕ_1 and double line represents ϕ_2. Since the divergent parts of these diagrams are all independent of external momenta, we can therefore simply ignore them. Then it's easy to evaluate them, as follows.



$$= \frac{(-i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{k^2} \sim \frac{i\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon}, \tag{12.16}$$

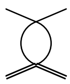
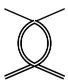


$$= \frac{(-i\rho/3)^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{k^2} \sim \frac{i\rho^2}{18(4\pi)^2}\frac{2}{\epsilon}. \tag{12.17}$$

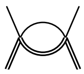
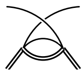
The t-channel and u-channel give the same result. Thus we can determine δ_λ to be

$$\delta_\lambda \sim \frac{9\lambda^2+\rho^2}{6(4\pi)^2}\frac{2}{\epsilon}. \tag{12.18}$$

On the other hand,

 = 

$$= \frac{(-i\lambda)(-i\rho/3)}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{k^2} \sim \frac{i\lambda\rho}{6(4\pi)^2}\frac{2}{\epsilon}, \tag{12.19}$$

 = 

$$= (-i\rho/3)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{k^2} \sim \frac{i\rho^2}{9(4\pi)^2}\frac{2}{\epsilon}. \tag{12.20}$$

Then we have

$$\delta_\rho \sim \frac{3\lambda\rho+2\rho^2}{3(4\pi^2)}\frac{2}{\epsilon}. \tag{12.21}$$

It's easy to see that field strengths for both ϕ_1 and ϕ_2 receives no contributions from 1-loop diagrams. Thus the 1-loop beta functions can be evaluated as

$$\beta_\lambda = -\mu\frac{d\delta_\lambda}{d\mu} = \frac{3\lambda^2+\rho^2/3}{(4\pi)^2}; \tag{12.22}$$

$$\beta_\rho = -\mu\frac{d\delta_\rho}{d\mu} = \frac{2\lambda\rho+4\rho^2/3}{(4\pi)^2}. \tag{12.23}$$

(b) Now we derive the renormalization equation for ρ/λ:

$$\mu\frac{d}{d\mu}\Big(\frac{\rho}{\lambda}\Big) = \frac{1}{\lambda}\beta_\rho - \frac{\rho}{\lambda^2}\beta_\lambda = \frac{\rho}{3(4\pi)^2}\Big[-(\rho/\lambda)^2+4(\rho/\lambda)-3\Big]. \tag{12.24}$$

Then it is easy to see that $\rho/\lambda=1$ is an IR fixed point.

(c) In $4-\epsilon$ dimensions, the β functions for ρ and λ are shifted as

$$\beta_\lambda = -\epsilon\lambda + \frac{3\lambda^2+\rho^2/3}{(4\pi)^2}; \tag{12.25}$$

$$\beta_\rho = -\epsilon\rho + \frac{2\lambda\rho+4\rho^2/3}{(4\pi)^2}. \tag{12.26}$$

But it is easy to show that the terms containing ϵ cancel out in the β function for ρ/λ, and the result is the same as (12.24). This is true because ρ/λ still remains dimensionless in $4-\epsilon$ dimensions. Therefore we conclude that there are three fixed points of the RG flow for ρ/λ at 0, 1, and 3. We illustrate this in the diagram of RG flow in the ρ-λ plane, with the deviation of dimension $\epsilon = 0.01$, in Figure 12.1.

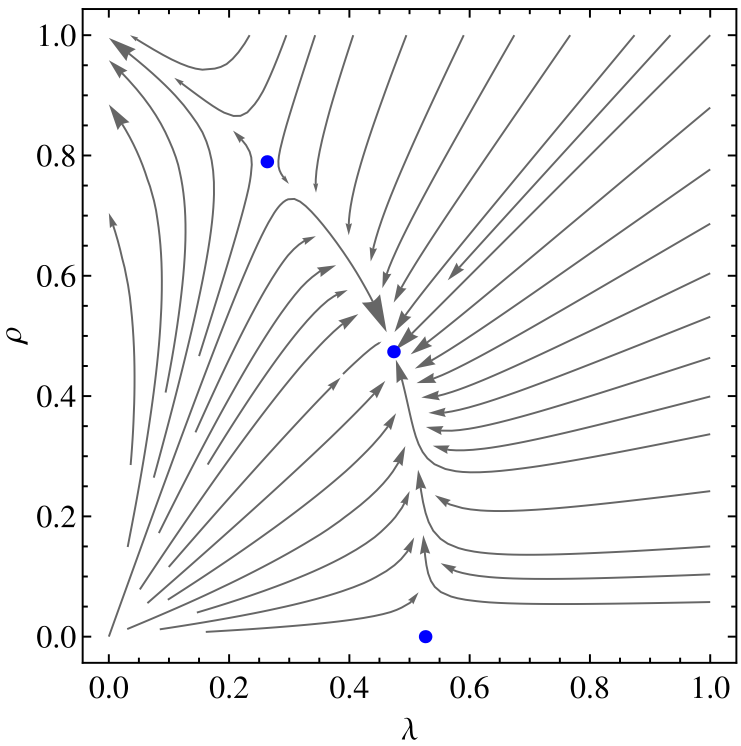


Figure 12.1: The RG flow of the theory (12.15) in $4 - \epsilon$ dimensions with $\epsilon = 0.01$. Three nontrivial fixed points are shown by blue dots.

Chapter 13

Critical Exponents and Scalar Field Theory

13.1 Correlation-to-scaling exponent

In this problem we consider the effect of the deviation of the coupling λ from its fixed point to the two-point correlation function $G(M,t)$ in $d = 4-\epsilon$ dimensions. Symbolically, we can always write

$$G(M,t) = G_*(M,t) + \left.\frac{\delta G(M,t)}{\delta\bar{\lambda}}\right|_{\bar{\lambda}=\lambda_*}\delta\bar{\lambda}, \tag{13.1}$$

where $\bar{\lambda}$ is the running coupling, defined to be the solution of the following renormalization group equation:

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\bar{\lambda} = \frac{2\beta_\lambda(\bar{\lambda})}{d-2+2\gamma(\bar{\lambda})}. \tag{13.2}$$

As the first step, let us expand the β function of $\bar{\lambda}$ around the fixed point, as

$$\begin{aligned}\beta(\bar{\lambda}) &= \beta(\lambda_*) + \left.\frac{\mathrm{d}\beta(\bar{\lambda})}{\mathrm{d}\bar{\lambda}}\right|_{\bar{\lambda}=\lambda_*}(\bar{\lambda}-\lambda_*) + \mathcal{O}((\bar{\lambda}-\lambda_*)^2)\\ &= \omega(\bar{\lambda}-\lambda_*) + \mathcal{O}((\bar{\lambda}-\lambda_*)^2).\end{aligned} \tag{13.3}$$

Then the renormalization group equation reads

$$\frac{\mathrm{d}}{\mathrm{d}\log\mu}\bar{\lambda} \simeq \frac{2\omega(\bar{\lambda}-\lambda_*)}{d-2+2\gamma(\lambda_*)} = \frac{\omega\nu}{\beta}(\bar{\lambda}-\lambda_*), \tag{13.4}$$

where β and ν on the right hand side are critical exponents, which in our case are defined to be

$$\beta = \frac{d-2+2\gamma(\lambda_*)}{d(2-\gamma_{\phi^*}(\lambda_*))}, \qquad \nu = \frac{1}{2-\gamma_{\phi^2}(\lambda_*)}.$$

Don't confuse the critical exponent β with the β function. Now, from this equation we can solve the running coupling $\bar{\lambda}$ to be

$$\bar{\lambda} = \lambda_* + \big(\bar{\lambda}(\mu_0)-\lambda_*\big)\Big(\frac{\mu}{\mu_0}\Big)^{\omega\nu/\beta}. \tag{13.5}$$

Now let μ_0 be the scale at which the bare coupling is defined. Then we get

$$\delta\bar{\lambda} \propto (\lambda - \lambda_*)\mu^{\omega\nu/\beta}. \tag{13.6}$$

13.2 The exponent η

We have found the counterterm δ_Z to $\mathcal{O}(\lambda^2)$ with $\overline{MS}$ scheme in Problem 10.3, to be

$$\delta_Z = -\frac{\lambda^2}{12(4\pi)^4}\left[\frac{1}{\epsilon} - \log M^2\right]. \tag{13.7}$$

Then the anomalous dimension γ to $\mathcal{O}(\lambda^2)$ is given by

$$\gamma = \frac{1}{2}M\frac{\partial}{\partial M}\delta_Z = \frac{\lambda^2}{12(4\pi)^4}. \tag{13.8}$$

This result can be easily generalized to the $O(N)$-symmetric ϕ^4 theory, by replacing the Feynman rule of the ϕ^4 coupling $-\mathrm{i}\lambda$ with

$$-2\mathrm{i}\lambda(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{j\ell} + \delta^{i\ell}\delta^{jk}),$$

which is equivalent to multiplying the diagram (10.19) by the following factor:

$$4 \cdot \big(\delta^{ik}\delta^{\ell m} + \delta^{i\ell}\delta^{km} + \delta^{im}\delta^{k\ell}\big)\big(\delta^{jk}\delta^{\ell m} + \delta^{j\ell}\delta^{km} + \delta^{jm}\delta^{k\ell}\big) = 12(N+2)\delta^{ij}, \tag{13.9}$$

and the anomalous dimension (13.8) obtained above should be multiplied by $12(N+2)$, which leads to

$$\gamma = (N+2)\frac{\lambda^2}{(4\pi)^4}, \tag{13.10}$$

which is the same as (13.47) of Peskin&Schroeder.

13.3 The CP^N model

(a) The Lagrangian of the CP^N model can be written as

$$\mathcal{L} = \frac{1}{g^2}\bigg(\sum_j |\partial_\mu z_j|^2 - \Big|\sum_j z_j^*\partial_\mu z_j\Big|^2\bigg), \tag{13.11}$$

with z_j $(j = 1, \cdots N+1)$ the components of a vector in $(N+1)$ dimensional complex space, subject to the constraint

$$\sum_j |z_j|^2 = 1 \tag{13.12}$$

and the identification

$$(e^{\mathrm{i}\alpha}z_1, \cdots, e^{\mathrm{i}\alpha}z_{N+1}) \sim (z_1, \cdots, z_{N+1}). \tag{13.13}$$

Now we prove that the Lagrangian given above is invariant under the following local transformation:

$$z_j(x) \to e^{\mathrm{i}\alpha(x)} z_j(x), \tag{13.14}$$

as,

$$\begin{aligned} g^2\mathcal{L} \to & \left|\partial_\mu(e^{\mathrm{i}\alpha} z_j)\right|^2 + \left|e^{-\mathrm{i}\alpha} z_j^* \partial_\mu(e^{\mathrm{i}\alpha} z_j)\right|^2 \\ =& \Big(|\partial_\mu z_j|^2 + |\partial_\mu \alpha|^2 + 2\mathrm{Re}\big(-\mathrm{i}(\partial_\mu\alpha) z_j^* \partial_\mu z_j\big)\Big) \\ & - \Big(|z_j^*\partial_\mu z_j|^2 + |\partial_\mu\alpha|^2 + 2\mathrm{Re}\big(-\mathrm{i}(\partial_\mu\alpha) z_i z_i^* z_j^* \partial_\mu z_j\big)\Big) \\ =& g^2\mathcal{L}. \end{aligned} \tag{13.15}$$

Then we show that the nonlinear σ model with $n = 3$ is equivalent to the CP^N model with $N = 1$. To see this, we substitute $n^i = z^*\sigma^i z$ into the Lagrangian of the nonlinear sigma model, $\mathcal{L} = \frac{1}{2g^2}|\partial_\mu n^i|^2$, to get

$$\begin{aligned} \mathcal{L} =& \frac{1}{2g^2}\Big|(\partial_\mu z^*)\sigma^i z + z^*\sigma^i\partial_\mu z\Big|^2 \\ =& \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z)z^* z + (\partial_\mu z)^2 z^{*2} + (\partial_\mu z^*)^2 z^2\Big] \\ =& \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z) + \big(z^*\partial_\mu z + z\partial_\mu z^*\big)^2 - 2(z^*\partial_\mu z)(z\partial^\mu z^*)\Big] \\ =& \frac{1}{2g^2}\sigma^i\sigma^i\Big[2(\partial_\mu z^*)(\partial^\mu z) + \big[\partial_\mu(z^* z)\big]^2 - 2(z^*\partial_\mu z)(z\partial^\mu z^*)\Big]. \end{aligned} \tag{13.16}$$

Then after a proper normalization of the field z, it is straightforward to see that the Lagrangian above reduces to

$$\mathcal{L} = \frac{1}{g^2}\Big(|\partial_\mu z|^2 - 2|z^*\partial_\mu z|^2\Big), \tag{13.17}$$

which is indeed the CP^1 model.

(b) The Lagrangian (13.11) can be obtained by the following Lagrangian with a gauge field A_μ and a Lagrange multiplier which expresses the local gauge symmetry and the constraint explicitly:

$$\mathcal{L} = \frac{1}{g^2}\Big(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2 - 1\big)\Big), \tag{13.18}$$

with $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}A_\mu$. Now let us verify this by functionally integrating out the gauge field A_μ as well as the Lagrange multiplier λ to get

$$\begin{aligned} Z =& \int \mathcal{D}^2 z_i \mathcal{D}A_\mu \mathcal{D}\lambda \,\exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x\,\Big(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2-1\big)\Big)\right] \\ =& \int \mathcal{D}^2 z_i \mathcal{D}A_\mu\, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x\, |\mathrm{D}_\mu z_j|^2\right] \\ =& \int \mathcal{D}^2 z_i \mathcal{D}A_\mu\, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x\,\Big(A_\mu A^\mu + 2\mathrm{i}A^\mu(\partial_\mu z_j^*)z_j + |\partial_\mu z_j|^2\Big)\right] \\ =& N\int \mathcal{D}^2 z_i\, \delta\big(|z_j|^2 - 1\big) \exp\left[\frac{\mathrm{i}}{g^2}\int \mathrm{d}^2 x\,\Big(|\partial_\mu z_j|^2 - |z_j^*\partial_\mu z_j|\Big)^2\right]. \end{aligned} \tag{13.19}$$

(c) On the other hand one can also integrate out z_i field in the Lagrangian (13.18), as

$$\begin{aligned} Z &= \int \mathcal{D}z_i \mathcal{D}A_\mu \mathcal{D}\lambda \, \exp\left[\frac{\mathrm{i}}{g^2} \int \mathrm{d}^2 x \left(|\mathrm{D}_\mu z_j|^2 - \lambda\big(|z_j|^2 - 1\big) \right) \right] \\ &= \int \mathcal{D}A_\mu \mathcal{D}\lambda \, \exp\left[- N \,\mathrm{tr}\, \log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2} \int \mathrm{d}^2 x \, \lambda \right] \end{aligned} \tag{13.20}$$

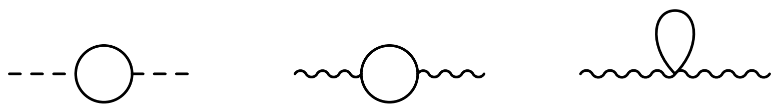
We assume that the expectation values for A_μ and λ are constants. Then the exponent can be evaluated by means of dimensional regularization, as

$$\begin{aligned} \mathrm{i}S = & - N \,\mathrm{tr}\, \log(-D^2 - \lambda) + \frac{\mathrm{i}}{g^2} \int \mathrm{d}^2 x \, \lambda \\ = & \left[- N \int \frac{\mathrm{d}^d k}{(2\pi)^d} \log\left(k^2 + A_\mu A^\mu - \lambda\right) + \frac{\mathrm{i}}{g^2}\lambda \right] \cdot V^{(2)} \\ \Rightarrow & \ \mathrm{i}\left[- \frac{N}{4\pi} \Big(\log \frac{M^2}{\lambda - A^2} + 1 \Big)(\lambda - A^2) + \frac{1}{g^2}\lambda \right] \cdot V^{(2)}, \end{aligned} \tag{13.21}$$

where $V^{(2)} = \int \mathrm{d}^2 x$, and we have used the $\overline{MS}$ scheme to subtract the divergence. Now we can minimize the quantity in the square bracket in the last line to get

$$A_\mu = 0, \qquad \lambda = M^2 \exp\Big(- \frac{4\pi}{gN^2} \Big). \tag{13.22}$$

(d) The meaning of the effective action S is most easily seen from its diagrammatic representations. For instance, at the 1-loop level, we know that the logarithmic terms in the effective action is simply the sum of a series of 1-loop diagrams with $n \geq 0$ external legs, where the number of external legs n is simply the power of corresponding fields in the expansion of S. Therefore, to the second order in A and in λ, the effective action is represented precisely by the following set of diagrams,



where the dashed lines represent λ, curved lines represent A_μ, and the internal loop are z field. Then it is straightforward to see that the correct kinetic terms for λ and A_μ are generated from these diagrams. That is, the gauge field A_μ becomes dynamical due to quantum corrections. The gauge invariance of the resulted kinetic term $F_{\mu\nu}F^{\mu\nu}$ can also be justified by explicit calculation as was done in Problem 9.1.

Final Project II
The Coleman-Weinberg Potential

In this final project, we work out some properties of Coleman-Weinberg model, illustrating basic techniques of the renormalization group. The original paper [6] by S. Coleman and E. Weinberg is always a good read, while a recent and very insightful treatment of the model can be found in [7].

Simply put, the Coleman-Weinberg model is a theory of scalar electrodynamics, described by the Lagrangian,

$$\mathcal{L} = -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi) - m^2\phi^\dagger\phi - \tfrac{\lambda}{6}(\phi^\dagger\phi)^2, \tag{13.23}$$

with ϕ a complex scalar and $D_\mu\phi = (\partial_\mu + \mathrm{i}eA_\mu)\phi$.

(a) Consider the case of spontaneous breaking of the $U(1)$ gauge symmetry $\phi(x) \to e^{\mathrm{i}\alpha(x)}\phi(x)$, caused by a negative squared mass, namely $m^2 = -\mu^2 < 0$. The scalar then acquires a nonzero vacuum expectation value (VEV) $\phi_0 = \sqrt{\langle|\phi|^2\rangle}$. We split this VEV out of the scalar field, namely,

$$\phi = \phi_0 + \frac{1}{\sqrt{2}}\big[\sigma(x) + \mathrm{i}\pi(x)\big], \tag{13.24}$$

with the new field $\sigma(x)$ and $\pi(x)$ being real. At the tree level, it is easy to find $\phi_0 = \sqrt{3\mu^2/\lambda}$ by minimize the scalar potential $V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{6}(\phi^\dagger\phi)^2$. We also introduce $v = \sqrt{2}\phi_0$ for convenience. Then, rewrite the Lagrangian in terms of these new field variables, we get,

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(F_{\mu\nu})^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\pi)^2 + \tfrac{1}{2}e^2v^2A_\mu A^\mu - \tfrac{1}{2}(2\mu^2)\sigma^2\\ &- \tfrac{\lambda}{24}(\pi^4 + \sigma^4 + 2\pi^2\sigma^2 + 4v\pi^2\sigma + 4v\sigma^3) + evA_\mu\partial^\mu\pi\\ &+ eA_\mu(\sigma\partial^\mu\pi - \pi\partial^\mu\sigma) + \tfrac{1}{2}e^2A_\mu A^\mu(\pi^2 + \sigma^2 + 2v\sigma).\end{aligned} \tag{13.25}$$

Then we see that the vector field A_μ acquires a mass, equal to $m_A = ev$ at the classical level.

(b) Now we calculate the 1-loop effective potential of the model. We know that 1-loop correction of the effective Lagrangian is given by,

$$\Delta\mathcal{L} = \frac{\mathrm{i}}{2}\log\det\left[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\right]_{\varphi=0} + \delta\mathcal{L}, \tag{13.26}$$

where φ is the fluctuating fields and $\delta\mathcal{L}$ denotes counterterms.

Let the background value of the complex scalar be $\phi_{\rm cl}$. By the assumption of Poincaré symmetry, $\phi_{\rm cl}$ must be a constant. For the same reason, the background value of the vector field A_μ must vanish. In addition, we can set $\phi_{\rm cl}$ to be real without loss of generality. Then we have,

$$\phi(x) = \phi_{\rm cl} + \varphi_1(x) + {\rm i}\varphi_2(x),$$

where $\varphi_1(x)$, $\varphi_2(x)$, together with $A_\mu(x)$, now serve as fluctuating fields. Expanding the Lagrangian around the background fields and keeping terms quadratic in fluctuating fields only, we get,

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} + \big|(\partial_\mu + {\rm i}eA_\mu)(\phi_{\rm cl}+\varphi_1+{\rm i}\varphi_2)\big|^2 \\ & - m^2\big|\phi_{\rm cl}+\varphi_1+{\rm i}\varphi_2\big|^2 - \tfrac{\lambda}{6}\big|\phi_{\rm cl}+\varphi_1+{\rm i}\varphi_2\big|^4 \\ =& \ \tfrac{1}{2}A_\mu\big[g^{\mu\nu}(\partial^2+2e^2\phi_{\rm cl}^2) - \partial^\mu\partial^\nu\big]A_\nu + \tfrac{1}{2}\varphi_1\big(-\partial^2 - m^2 - \lambda\phi_{\rm cl}^2\big)\varphi_1 \\ & + \tfrac{1}{2}\varphi_2\big(-\partial^2 - m^2 - \tfrac{\lambda}{3}\phi_{\rm cl}^2\big)\varphi_2 - 2e\phi_{\rm cl}A_\mu\partial^\mu\varphi_2 + \cdots\,, \end{aligned} \tag{13.27}$$

where "$\cdots$" denotes terms other than being quadratic in fluctuating fields. Now we impose the Landau gauge condition $\partial_\mu A^\mu = 0$ to the Lagrangian, which removes the off-diagonal term $-2e\phi_{\rm cl}A_\mu\partial^\mu\varphi_2$. Then, according to (13.26), the 1-loop effective Lagrangian can be evaluated as,

$$\begin{aligned}\frac{\rm i}{2}\log\det\Big[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\Big]_{\varphi=0} =& \frac{\rm i}{2}\Big[\log\det\big(-\eta^{\mu\nu}(\partial^2+2e^2\phi_{\rm cl}^2)+\partial^\mu\partial^\nu\big) \\ & + \log\det\big(\partial^2+m^2+\lambda\phi_{\rm cl}^2\big) + \log\det\big(\partial^2+m^2+\tfrac{\lambda}{3}\phi_{\rm cl}^2\big)\Big] \\ =& \frac{\rm i}{2}\int\frac{{\rm d}^dk}{(2\pi)^d}\Big[{\rm tr}\ \log(-k^2+2e^2\phi_{\rm cl}^2)^3 \\ & + {\rm tr}\ \log(-k^2+m^2+\lambda\phi_{\rm cl}^2) + {\rm tr}\ \log(-k^2+m^2+\tfrac{\lambda}{3}\phi_{\rm cl}^2)\Big] \\ =& \frac{\Gamma(-\frac{d}{2})}{2(4\pi)^{d/2}}\Big[3(2e^2\phi_{\rm cl}^2)^{d/2} + (m^2+\lambda\phi_{\rm cl}^2)^{d/2} + (m^2+\tfrac{\lambda}{3}\phi_{\rm cl}^2)^{d/2}\Big]. \end{aligned} \tag{13.28}$$

In the second equality we use the following identity,

$$\det(\lambda I + AB) = \lambda^{n-1}(\lambda + BA), \tag{13.29}$$

where A and B are matrices of $n\times 1$ and $1\times n$, respectively, λ is an arbitrary complex number and I is the $n\times n$ identity matrix. In our case, this gives,

$$\det\big(-\eta^{\mu\nu}(\partial^2+2e^2\phi_{\rm cl}^2)+\partial^\mu\partial^\nu\big) = -2e^2\phi_{\rm cl}^2(\partial^2+2e^2\phi_{\rm cl}^2)^3. \tag{13.30}$$

Then the second equality follows up to an irrelevant constant term. The third equality makes use of the trick in (11.72) of P&S. Then, for $d = 4-\epsilon$ and $\epsilon\to 0$, we have,

$$\frac{\rm i}{2}\log\det\Big[-\frac{\delta^2\mathcal{L}}{\delta\varphi\delta\varphi}\Big]_{\varphi=0} = \frac{1}{4(4\pi)^2}\Big[3(2e^2\phi_{\rm cl}^2)^2\big(\Delta - \log(2e^2\phi_{\rm cl}^2)\big)$$

$$+ (m^2 + \lambda\phi_{\text{cl}}^2)^2\big(\Delta - \log(m^2 + \lambda\phi_{\text{cl}}^2)\big)$$
$$+ (m^2 + \tfrac{\lambda}{3}\phi_{\text{cl}}^2)^2\big(\Delta - \log(m^2 + \tfrac{\lambda}{3}\phi_{\text{cl}}^2)\big)\Big], \quad (13.31)$$

where we define $\Delta \equiv \frac{2}{\epsilon} - \gamma + \log 4\pi + \frac{3}{2}$ for brevity.

Now, with $\overline{MS}$ scheme, we can determine the counterterms in (13.26) to be

$$\delta\mathcal{L} = \frac{-1}{4(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \log 4\pi - \log M^2\right]\Big(3(2e^2\phi_{\text{cl}}^2)^2 + (m^2 + \lambda\phi_{\text{cl}}^2)^2 + (m^2 + \tfrac{\lambda}{3}\phi_{\text{cl}}^2)^2\Big). \quad (13.32)$$

where M is the renormalization scale. Now the effective potential follows directly from (13.26), (13.31) and (13.32),

$$V_{\text{eff}}[\phi_{\text{cl}}] = m^2\phi_{\text{cl}}^2 + \frac{\lambda}{6}\phi_{\text{cl}}^4 - \frac{1}{4(4\pi)^2}\bigg[3(2e^2\phi_{\text{cl}}^2)^2\Big(\log\frac{M^2}{2e^2\phi_{\text{cl}}^2} + \frac{3}{2}\Big)$$
$$+ (m^2 + \lambda\phi_{\text{cl}}^2)^2\Big(\log\frac{M^2}{m^2 + \lambda\phi_{\text{cl}}^2} + \frac{3}{2}\Big) + (m^2 + \tfrac{\lambda}{3}\phi_{\text{cl}}^2)^2\Big(\log\frac{M^2}{m^2 + \frac{\lambda}{3}\phi_{\text{cl}}^2} + \frac{3}{2}\Big)\bigg]. \quad (13.33)$$

(c) Now taking the mass parameter $\mu^2 = -m^2 = 0$, then the effective potential (13.33) becomes

$$V_{\text{eff}}[\phi_{\text{cl}}] = \frac{\lambda}{6}\phi_{\text{cl}}^4 + \frac{1}{4(4\pi)^2}\bigg[12e^4\phi_{\text{cl}}^4\Big(\log\frac{2e^2\phi_{\text{cl}}^2}{M^2} - \frac{3}{2}\Big) + \frac{10}{9}\lambda^2\phi_{\text{cl}}^4\Big(\log\frac{\lambda\phi_{\text{cl}}^2}{M^2} - \frac{3}{2}\Big)\bigg]$$
$$\simeq \frac{\lambda}{6}\phi_{\text{cl}}^4 + \frac{3e^4\phi_{\text{cl}}^4}{(4\pi)^2}\Big(\log\frac{2e^2\phi_{\text{cl}}^2}{M^2} - \frac{3}{2}\Big). \quad (13.34)$$

In the second line we use the fact that λ is of the order e^4 to drop the λ^2 term. Then the minimal point of this effective potential can be easily worked out to be,

$$\phi_{\text{cl}}^2 = \frac{M^2}{2e^2}\exp\Big(1 - \frac{8\pi^2\lambda}{9e^4}\Big). \quad (13.35)$$

As $\lambda \sim e^4$, we see that ϕ_{cl} is of the same order with $e^{-1}M$. Thus the effective potential remains valid at this level of perturbation theory.

(d) We plot the effective potential as a function of ϕ_{cl} in Figure (13.1). The purple curve with $m^2 = 5\times 10^{-7}M^2$ corresponds the case with no spontaneous symmetry breaking. The blue curve shows that as m^2 goes to 0 from above, new local minima is formed. Finally, the orange and red curves correspond to broken symmetry, and in the case of the orange curve with $m^2 = 0$, the symmetry is dynamically broken.

(e) Now we calculate β functions of the Coleman-Weinberg model to 1-loop level at high energies, where we can send the mass parameter m^2 to zero. It is convenient to work in the Feynman gauge $\xi = 1$. Then the relevant Feynman rules can be read from the Lagrangian (13.25) to be,

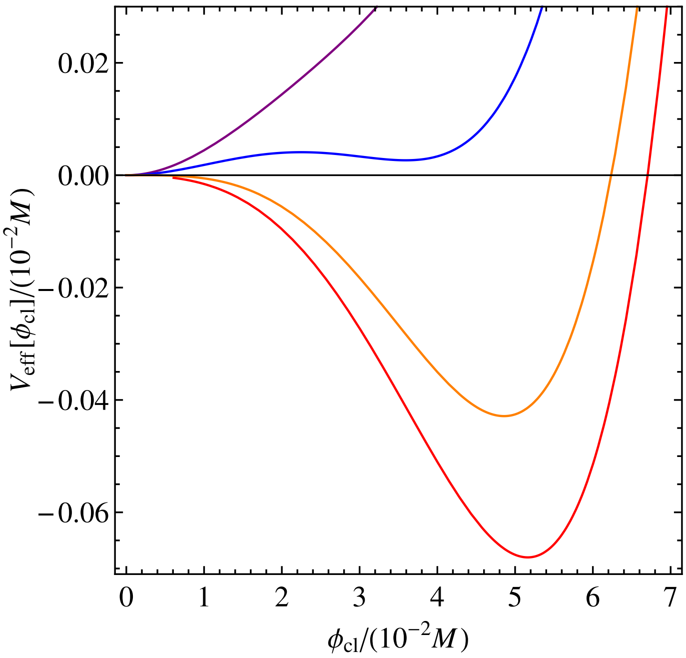
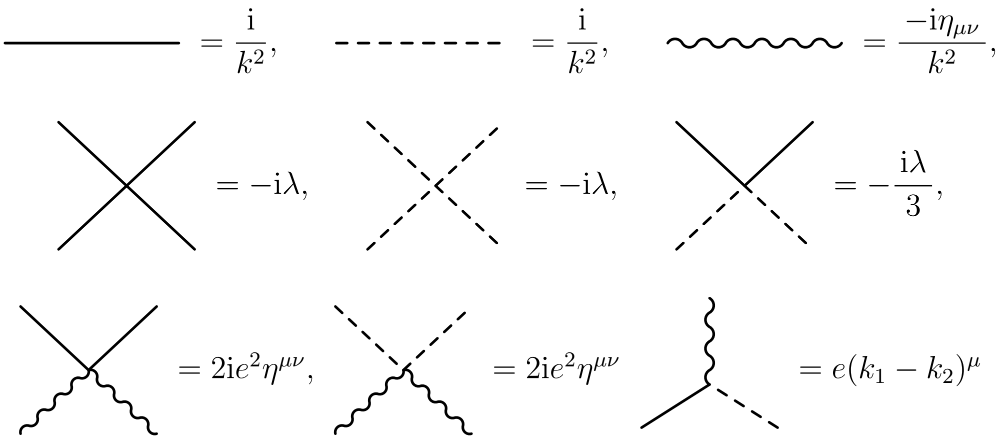
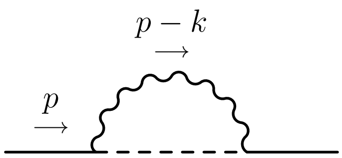


Figure 13.1: The effective potential V_{eff} as a function of ϕ_{cl}, with different values of $m^2/M^2 = 5\times10^{-7}$, 2.4×10^{-7}, 0 and -1×10^{-7} from top to bottom, respectively.



We first find the 1-loop wave function renormalization. For σ field, there is only one diagram with nonzero contribution,



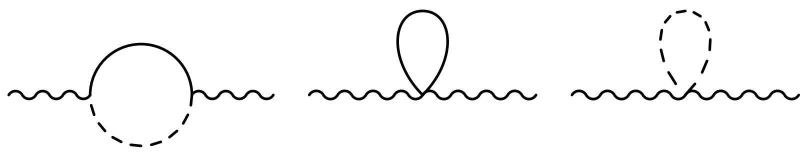
which reads,

$$e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(p-k)^2}(p+k)_\mu(-p-k)^\mu \sim -\frac{2\mathrm{i}e^2p^2}{(4\pi)^2}\frac{2}{\epsilon}. \tag{13.36}$$

Then we have,

$$\delta_\sigma = \frac{2e^2}{(4\pi)^2}\Big(\frac{2}{\epsilon}-M^2\Big), \tag{13.37}$$

and it is straightforward to see that $\delta_\pi = \delta_\sigma$. For photon's wave function renormalization (vacuum polarization), we need to evaluate the following three diagrams,



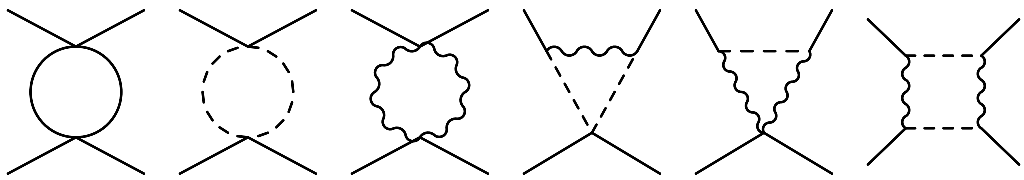
The sum of the three diagrams is,

$$\begin{aligned}&-e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p-k)^2}(2k-p)_\mu(2k-p)_\nu+2\cdot\frac{1}{2}\cdot 2\mathrm{i}e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\cdot\frac{(p-k)^2}{(p-k)^2}\\&\sim-\frac{\mathrm{i}e^2}{3(4\pi)^2}\frac{2}{\epsilon}(p^2\eta_{\mu\nu}-p_\mu p_\nu),\end{aligned}\tag{13.38}$$

which gives,

$$\delta_A=-\frac{e^2}{3(4\pi)^2}\Big(\frac{2}{\epsilon}-\log M^2\Big).\tag{13.39}$$

Then we turn to the 1-loop corrections to couplings. For scalar self-coupling λ, we consider the 1-loop corrections to σ^4 term in the Lagrangian. There are six types of diagrams contributing, listed as follows, and we label them by (a) to (f) from left to right,



For each type there are several different permutations of internal lines giving identical result, or more concretely, 3 permutations for each of the first three types, and 6 permutations for each of the last three types. Now we evaluate them in turn. We set all external momenta to zero to simplify the calculation. Then,

$$(\mathrm{a})=\frac{(-\mathrm{i}\lambda)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\sim\frac{\mathrm{i}\lambda^2}{2(4\pi)^2}\frac{2}{\epsilon},\tag{13.40}$$

$$(\mathrm{b})=\frac{(-\mathrm{i}\lambda/3)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\sim\frac{\mathrm{i}\lambda^2}{18(4\pi)^2}\frac{2}{\epsilon},\tag{13.41}$$

$$(\mathrm{c})=\frac{(2\mathrm{i}e^2)^2}{2}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2\eta_{\mu\nu}\eta^{\mu\nu}\sim\frac{8\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon},\tag{13.42}$$

$$(\mathrm{d})=\frac{-\mathrm{i}\lambda e^2}{3}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\Big(\frac{\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)\sim-\frac{\mathrm{i}\lambda e^2}{3(4\pi)^2}\frac{2}{\epsilon},\tag{13.43}$$

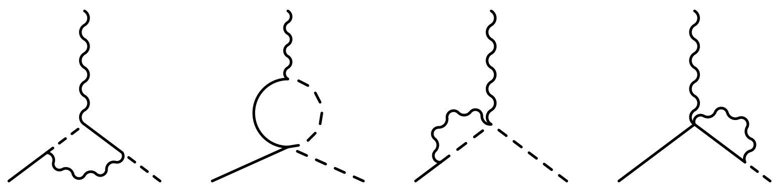
$$(\mathrm{e})=(2\mathrm{i}e^2)e^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)\sim-\frac{2\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon},\tag{13.44}$$

$$(\mathrm{f})=e^4\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2\Big(\frac{-\mathrm{i}}{k^2}\Big)^2(-k_\mu k^\mu)^2\sim\frac{\mathrm{i}e^4}{(4\pi)^2}\frac{2}{\epsilon},\tag{13.45}$$

Then multiplying (a)$\sim$(c) by 3 and (d)$\sim$(f) by 6, we find

$$\delta_\lambda = \frac{5\lambda^2/3 - 2\lambda e^2 + 18e^4}{(4\pi)^2}\frac{2}{\epsilon}. \tag{13.46}$$

Finally we consider the 1-loop corrections to e. For this purpose we calculate 1-loop diagrams with three external lines with 1 A_μ, 1 σ and 1 π respectively, shown as follows, labeled again by (a) to (d) from left to right,



Now we calculate them in turn.

$$\begin{aligned}
(\text{a}) &= e^3 \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\Big(\frac{\mathrm{i}}{(p-k)^2}\Big)^2 (2p-k)^2(2k-2p)^\mu \\
&= \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{4x[(2-x)p-k']^2[k'-(1-x)p]^\mu}{[k'^2+x(1-x)p^2]^3} \\
&\sim \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{-k'^2(1-x)p^\mu - 2(2-x)(p\cdot k')k'^\mu}{[k'^2+x(1-x)p^2]^3} \\
&= \mathrm{i}e^3 \int \frac{\mathrm{d}^d k'}{(2\pi)^d}\int_0^1 \mathrm{d}x\, \frac{[-(1-x)-\frac{2}{d}(2-x)]k'^2p^\mu}{[k'^2+x(1-x)p^2]^3} \\
&\sim \frac{2e^3}{(4\pi)^2}\frac{2}{\epsilon}p^\mu,
\end{aligned} \tag{13.47}$$

$$(\text{b}) = e\Big(\frac{-\mathrm{i}\lambda}{3}\Big)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\Big(\frac{\mathrm{i}}{k^2}\Big)^2 \cdot 2k^\mu = 0, \tag{13.48}$$

$$(\text{c}) = (\text{d}) = e(2\mathrm{i}e^2)\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(p-k)^2}(k+p)^\mu \sim -\frac{3e^3}{(4\pi)^2}\frac{2}{\epsilon}p^\mu. \tag{13.49}$$

Summing the four diagrams, we find that

$$\delta_e = \frac{2e^3}{(4\pi)^2}\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{13.50}$$

Now we are ready to calculate β functions,

$$\beta_e = M\frac{\partial}{\partial M}\Big(-\delta_e + \frac{1}{2}(\delta_A + \delta_\sigma + \delta_\pi)\Big) = \frac{e^3}{48\pi^2}, \tag{13.51}$$

$$\beta_\lambda = M\frac{\partial}{\partial M}\big(-\delta_\lambda + 2\delta_\sigma\big) = \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24\pi^2}. \tag{13.52}$$

The trajectory of renormalization group flows generated from these β functions are shown in Figure 13.2.

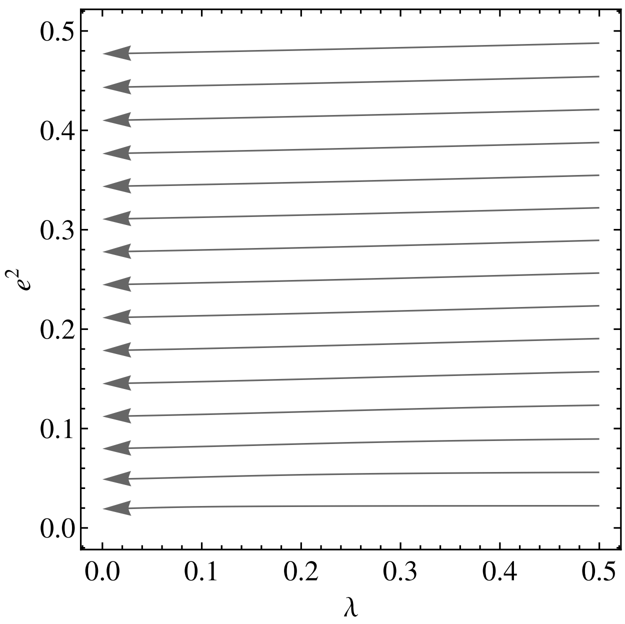


Figure 13.2: The renormalization group flow of Coleman-Weinberg model.

(f) The effective potential obtained in (c) is not a solution to the renormalization group equation, since it is only a first order result in perturbation expansion. However, it is possible to find an effective potential as a solution to the RG equation, with the result in (c) serving as a sort of "initial condition". The effective potential obtained in this way is said to be RG improved.

The Callan-Symansik equation for the effective potential reads

$$\left(M\frac{\partial}{\partial M}+\beta_\lambda\frac{\partial}{\partial\lambda}+\beta_e\frac{\partial}{\partial e}-\gamma_\phi\phi_{\rm cl}\frac{\partial}{\partial\phi_{\rm cl}}\right)V_{\rm eff}(\phi_{\rm cl},\lambda,e;M)=0. \tag{13.53}$$

The solution to this equation is well known, that is, the dependence of the sliding energy scale M is described totally by running parameters,

$$V_{\rm eff}(\phi_{\rm cl},\lambda,e;M)=V_{\rm eff}\big(\bar\phi_{\rm cl}(M'),\bar\lambda(M'),\bar e(M');M'\big), \tag{13.54}$$

where barred quantities satisfy

$$M\frac{\partial\bar\lambda}{\partial M}=\beta_\lambda(\bar\lambda,\bar e),\qquad M\frac{\partial\bar e}{\partial M}=\beta_e(\bar\lambda,\bar e),\qquad M\frac{\partial\bar\phi_{\rm cl}}{\partial M}=-\gamma_\phi(\bar\lambda,\bar e)\bar\phi_{\rm cl}. \tag{13.55}$$

The RG-improved effective potential should be such that when expanded in terms of coupling constants λ and e, it will recover the result in (c) at the given order. For simplicity here we work under the assumption that $\lambda\sim e^4$, so that all terms of higher orders of coupling constants than λ and e^4 can be ignored. In this case, the perturbative calculation in (c) gives

$$V_{\rm eff}=\frac{\lambda}{6}\phi_{\rm cl}^4+\frac{3e^4\phi_{\rm cl}^4}{(4\pi)^2}\Big(\log\frac{2e^2\phi_{\rm cl}^2}{M^2}-\frac{3}{2}\Big). \tag{13.56}$$

Now we claim that the RG-improved edition of this result reads

$$V_{\text{eff}} = \frac{\bar{\lambda}}{6}\bar{\phi}_{\text{cl}}^4 + \frac{3\bar{e}^4\bar{\phi}_{\text{cl}}^4}{(4\pi)^2}\Big(\log 2\bar{e}^2 - \frac{3}{2}\Big). \tag{13.57}$$

To see this, we firstly solve the renormalization group equations (13.55),

$$\bar{\lambda}(M') = \bar{e}^4\bigg(\frac{\lambda}{e^4} + \frac{9}{4\pi^2}\log\frac{M'}{M}\bigg), \tag{13.58}$$

$$\bar{e}^2(M') = \frac{e^2}{1-(e^2/24\pi^2)\log(M'/M)}, \tag{13.59}$$

$$\bar{\phi}_{\text{cl}}(M') = \phi_{\text{cl}}\Big(\frac{M'}{M}\Big)^{2e^2/(4\pi)^2}, \tag{13.60}$$

where the unbarred quantities λ, e and ϕ_{cl} are evaluated at scale M. Now we substitute these results back into the RG-improved effective potential (13.57) and expand in terms of coupling constants. Then it is straightforward to see that the result recovers (13.56). To see the spontaneous symmetry breaking still occurs, we note that the running coupling $\bar{\lambda}(M')$ flows to negative value rapidly for small $M' = \phi_{\text{cl}}$, while $\bar{e}(M')$ changes mildly along the ϕ_{cl} scale, as can be seen directly from Figure 13.2. Therefore the the coefficient before ϕ_{cl}^4 is negative for small ϕ_{cl} and positive for large ϕ_{cl}. As a consequence, the minimum of this effective potential should be away from $\phi_{\text{cl}} = 0$, namely the $U(1)$ symmetry is spontaneously broken.

To find the scalar mass m_σ in this case (with $\mu = 0$), we calculate the second derivative of the effective potential V_{eff} with respect to ϕ_{cl}. Since the renormalization scale M can be arbitrarily chosen, we set it to be $M^2 = 2e^2\langle\phi_{\text{cl}}^2\rangle$ to simplify the calculation. Then the vanishing of the first derivative of V_{eff} at $\phi_{\text{cl}} = \langle\phi_{\text{cl}}\rangle$ implies that $\lambda = 9e^4/8\pi^2$. Insert this back to V_{eff} in (13.56), we find that

$$V_{\text{eff}} = \frac{3e^4\phi_{\text{cl}}^4}{16\pi^2}\bigg(\log\frac{\phi_{\text{cl}}^2}{\langle\phi_{\text{cl}}^2\rangle} - \frac{1}{2}\bigg). \tag{13.61}$$

Then, taking the second derivative of this expression with respect to ϕ_{cl}, we get the scalar mass $m_\sigma^2 = 3e^4\langle\phi_{\text{cl}}^2\rangle/4\pi^2 = 3e^4v^2/8\pi^2$. Recall that the gauge boson's mass m_A is given by $m_A = e^2v^2$ at the leading order, thus we conclude that $m_\sigma^2/m_A^2 = 3e^2/8\pi^2$ at the leading order in e^2.

(g) Now we consider the effect of finite mass, by adding a positive quadratic term into the effective potential (13.56). For simplicity we still work with $\lambda = 9e^4/8\pi^2$, which is always attainable without tuning. Then the effective potential reads,

$$V_{\text{eff}} = m_r^2\phi_{\text{cl}}^2 + \frac{3e^4\phi_{\text{cl}}^4}{(4\pi)^2}\Big(\log\frac{2e^2\phi_{\text{cl}}^2}{M^2} - \frac{1}{2}\Big), \tag{13.62}$$

in which the mass m_r is not identical to the bare mass parameter appeared in the classical Lagrangian, but has include 1-loop correction. As m_r increases from zero, the energy of symmetry breaking vacuum at $\langle\phi_{\text{cl}}\rangle \neq 0$ also increases, until it reaches zero when $m = m_c > 0$, and has the same vacuum energy with the symmetric vacuum $\langle\phi_{\text{cl}}\rangle = 0$. Then for $m > m_c$, there will be no stable symmetry breaking vacuum.

Using the effective potential above, we can find the position of the symmetry breaking vacuum by solving the equation $\partial V_{\text{eff}}/\partial\phi_{\text{cl}}\big|_{\phi_{\text{cl}}=\phi_v} = 0$, with the following solution,

$$\phi_v^2 = \frac{M^2}{2e^2}\exp\left[\mathrm{W}\Big(-\frac{(4\pi)^2 m_r^2}{3e^2M^2}\Big)\right], \tag{13.63}$$

in which $\mathrm{W}(z)$ is the Lambert W function, defined as the solution of $z = \mathrm{W}(z)e^{\mathrm{W}(z)}$. Then we can use the condition $V_{\text{eff}}(\phi_v) = 0$ to determine m_c to be,

$$m_c^2 = \frac{3e^2M^2}{32\pi^2\mathrm{e}^{1/2}}, \tag{13.64}$$

where the e in denominator is 2.718... and should not be confused with electric charge e.

Now we can evaluate the mass ratio $m_\sigma^2/m_A^2 = \frac{1}{2}V''_{\text{eff}}(\phi_v)/(e\phi_v)^2$, as a function of mass parameter m_r, to be,

$$m_\sigma^2/m_A^2 = \frac{3e^2}{8\pi^2}\left[1 + \mathrm{W}\Big(-\frac{(4\pi)^2 m_r^2}{3e^2M^2}\Big)\right]. \tag{13.65}$$

This is a monotonically decreasing function of m_r, and when $m_r = 0$, it recovers the previous result $3e^2/8\pi^2$. On the other hand, when $m_r = m_c$, the mass ratio m_σ^2/m_A^2 reaches its minimum value, given by $\frac{3e^2}{8\pi^2}[1 + \mathrm{W}(-\frac{1}{2\mathrm{e}})] = 3e^2/16\pi^2$, which is one half of the massless case.

(h) When the spacetime dimension is shifted from 4 as $d = 4 - \epsilon$, the β functions β_e and β_λ are also shifted to be

$$\beta_e = -\epsilon e + \frac{e^3}{48\pi^2}, \qquad \beta_\lambda = -\epsilon\lambda + \frac{5\lambda^2 - 18\lambda e^2 + 54e^4}{24\pi^2}. \tag{13.66}$$

We plot the corresponding RG flow diagrams for several choice of ϵ in Figure 13.3, where we also extrapolate the result to $\epsilon = 1$.

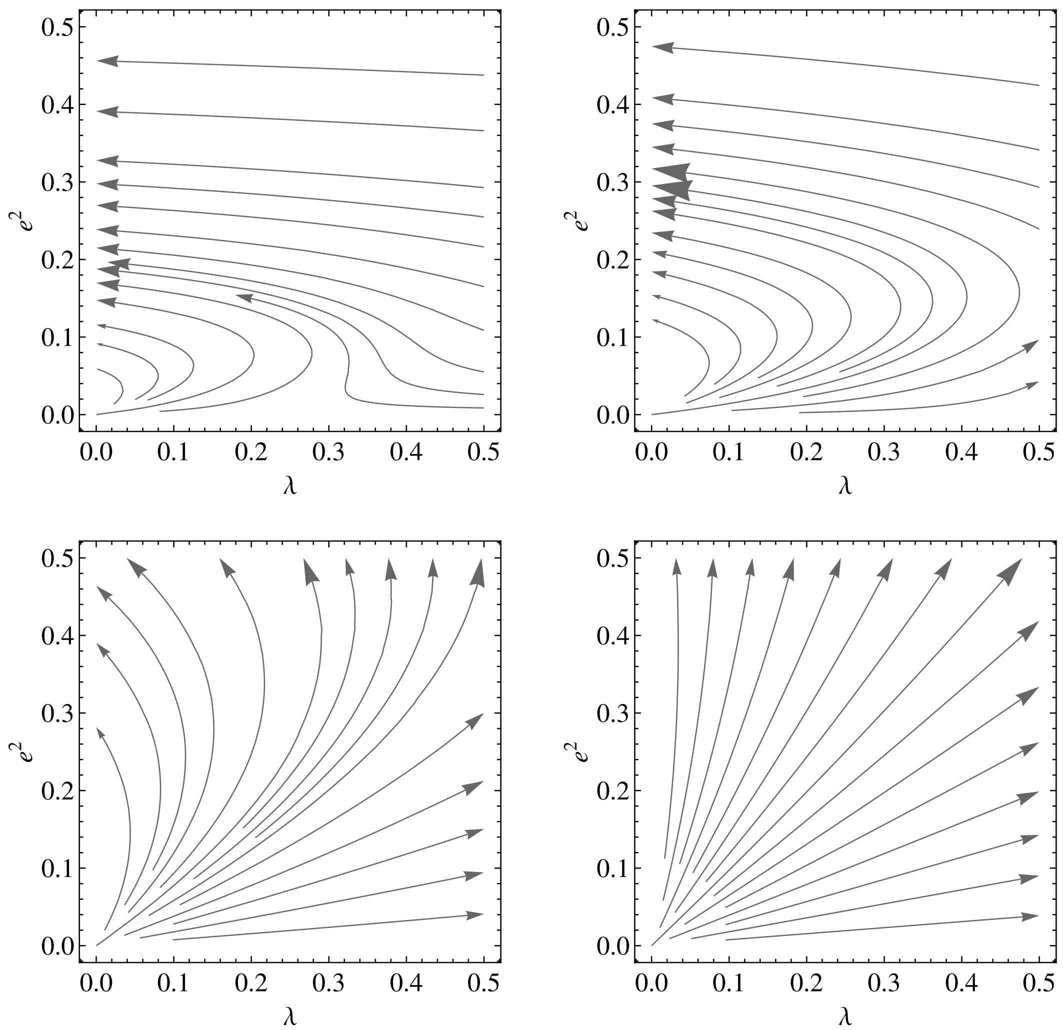


Figure 13.3: The renormalization group flows of Coleman-Weinberg model in $d = 4 - \epsilon$ spacetime dimensions, with $\epsilon = 0.005$, 0.01, 0.1 and 1 in the upper-left, upper-right, lower-left and lower-right diagram, respectively.

Chapter 15

Non-Abelian Gauge Invariance

15.1 Brute-force computations in $SU(3)$

(a) The dimension of $SU(N)$ group is $d = N^2 - 1$, when $N = 3$ we get $d = 8$.

(b) It's easy to see that t^1, t^2, t^3 generate a $SU(2)$ subgroup of $SU(3)$. Thus we have $f^{ijk} = \epsilon^{ijk}$ for $i, j, k = 1, 2, 3$. Just take another example, let's check $[t^6, t^7]$:

$$[t^6, t^7] = \mathrm{i}(-\tfrac{1}{2}t^3 + \tfrac{\sqrt{3}}{2}t^8),$$

thus we get

$$f^{678} = \tfrac{\sqrt{3}}{2}, \qquad f^{673} = -\tfrac{1}{2}.$$

Then what about f^{376}?

$$[t^3, t^7] = \tfrac{\mathrm{i}}{2}t^6 \qquad \Rightarrow \qquad f^{376} = \tfrac{1}{2} = -f^{673}.$$

(c) $C(F) = \frac{1}{2}$. Here F represents fundamental representation.

(d) $C_2(F) = \frac{4}{3}$, $d(F) = 3$, $d(G) = 8$, thus we see that $d(F)C_2(F) = d(G)C(F)$.

15.2 Adjoint representation of $SU(2)$

The structure constants for $SU(2)$ is $f^{abc} = \epsilon^{abc}$, thus we can write down the representation matrices for its generators directly from

$$(t^b_G)_{ac} = \mathrm{i}f^{abc} = \mathrm{i}\epsilon^{abc}.$$

More explicitly,

$$t^1_G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad t^2_G = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \qquad t^3_G = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15.1}$$

Then,

$$C(G) = \operatorname{tr}(t^1_G t^1_G) = \operatorname{tr}(t^2_G t^2_G) = \operatorname{tr}(t^3_G t^3_G) = 2,$$

$$C_2(G) I_3 = t^1_G t^1_G + t^2_G t^2_G + t^3_G t^3_G = 2 I_3 \quad \Rightarrow \quad C_2(G) = 2.$$

Here I_3 is the 3×3 unit matrix.

15.3 Coulomb potential

(a) We calculate vacuum expectation value for Wilson loop $U_P(z,z)$, defined by

$$U_P(z,z) = \exp\Big[-\mathrm{i}e\oint_P \mathrm{d}x^\mu A_\mu(x)\Big]. \tag{15.2}$$

By definition, we have

$$\langle U_P(z,z)\rangle = \int \mathcal{D}A_\mu \, \exp\Big[\mathrm{i}S[A_\mu] - \mathrm{i}e\oint_P \mathrm{d}x^\mu A_\mu(x)\Big], \tag{15.3}$$

where

$$S[A_\mu] = \int \mathrm{d}^4 x\,\Big[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\xi}(\partial^\mu A_\mu)^2\Big]. \tag{15.4}$$

Thus $\langle U_P(z,z)\rangle$ is simply a Gaussian integral, and can be worked out directly, as

$$\langle U_P(z,z)\rangle = \exp\Big[-\frac{1}{2}\big(-\mathrm{i}e\oint_P \mathrm{d}x^\mu\big)\big(-\mathrm{i}e\oint_P \mathrm{d}y^\nu\big)\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{-\mathrm{i}g_{\mu\nu}}{k^2+\mathrm{i}\epsilon}e^{-\mathrm{i}k\cdot(x-y)}\Big] \tag{15.5}$$

Here we have set $\xi\to 0$ to simplify the calculation. Working out the momentum integral, we get

$$\langle U_P(z,z)\rangle = \exp\Big[-\frac{e^2}{8\pi^2}\oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu \,\frac{g_{\mu\nu}}{(x-y)^2}\Big]. \tag{15.6}$$

The momentum integration goes as follows

$$\begin{aligned}
&\int\frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{e^{-\mathrm{i}k\cdot(x-y)}}{k^2+\mathrm{i}\epsilon} = \mathrm{i}\int\frac{\mathrm{d}^4 k_E}{(2\pi)^4}\frac{e^{\mathrm{i}k_E\cdot(x-y)}}{-k_E^2}\\
&= \frac{\mathrm{i}}{(2\pi)^4}\int_0^{2\pi}\mathrm{d}\psi\int_0^\pi \mathrm{d}\phi\sin\phi\int_0^\pi\mathrm{d}\theta\sin^2\theta\int_0^\infty \mathrm{d}k_E\,k_E^3\frac{e^{\mathrm{i}k_E|x-y|\cos\theta}}{-k_E^2}\\
&= -\frac{\mathrm{i}}{4\pi^3}\int_0^\infty \mathrm{d}k_E\,k_E\int_0^\pi \mathrm{d}\theta\sin^2\theta e^{\mathrm{i}k_E|x-y|\cos\theta}\\
&= -\frac{\mathrm{i}}{4\pi^2}\int_0^\infty \mathrm{d}k_E\,k_E\frac{J_1\big(k_E|x-y|\big)}{k_E|x-y|} = -\frac{\mathrm{i}}{4\pi^2(x-y)^2}.
\end{aligned} \tag{15.7}$$

Where $J_1(x)$ is Bessel function and we use the fact that $\int_0^\infty \mathrm{d}x\,J_1(x) = 1$.

(b) Now taking a narrow rectangular Wilson loop P with width R in x^1 direction ($0 < x^1 < 1$) and length T in x^0 direction ($0 < x^0 < T$) and evaluate $\langle U_P\rangle$. When the integral over $\mathrm{d}x$ and $\mathrm{d}y$ go independently over the loop, divergence will occur as $|x-y|^2 \to 0$. But what we want to show is the dependence of $\langle U_P\rangle$ on the geometry of the loop, namely the width R and length T, which should be divergence free. Therefore, when $T \gg R$, the integral in Wilson loop is mainly contributed by time direction and can be expressed as

$$\langle U_P(z,z)\rangle \simeq \exp\Big[-\frac{2e^2}{8\pi^2}\int_0^T \mathrm{d}x^0 \int_T^0 \mathrm{d}y^0 \frac{1}{(x^0-y^0)^2 - R^2 - \mathrm{i}\epsilon}\Big], \tag{15.8}$$

and we have add a small imaginary part to the denominator for the reason that will be clear. Carry out the integration, we find

$$\int_0^T \mathrm{d}x^0 \int_T^0 \mathrm{d}y^0 \frac{1}{(x^0-y^0)^2 - R^2 - \mathrm{i}\epsilon} \xrightarrow{T\gg R} \frac{2T}{R}\operatorname{arctanh}\Big(\frac{T}{R+\mathrm{i}\epsilon}\Big) = -\frac{\mathrm{i}}{\pi}\frac{T}{R}.$$

Therefore,

$$\langle U_P\rangle = \exp\Big(\frac{\mathrm{i}e^2}{4\pi R}\cdot T\Big) = e^{-\mathrm{i}V(R)T}, \tag{15.9}$$

which gives the familiar result $V(R) = -e^2/4\pi R$.

(c) For the Wilson loop of a non-Abelian gauge group, we have

$$U_P(z,z) = \operatorname{tr}\Big\{\mathrm{P}\exp\Big[-\mathrm{i}g\oint_P \mathrm{d}x^\mu A^a_\mu(x)t^a_r\Big]\Big\}, \tag{15.10}$$

where t^a_r is the matrices of the group generators in representation r. We expand this expression to the order of g^2,

$$\begin{aligned} U_P(z,z) =&\ \operatorname{tr}(1) - g^2\oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu A^a_\mu(x)A^b_\nu(y)\operatorname{tr}(t^a_r t^b_r) + \mathcal{O}(g^3)\\ =&\ \operatorname{tr}(1)\Big[1 - g^2 C_2(r)\oint_P \mathrm{d}x^\mu \oint_P \mathrm{d}y^\nu A^a_\mu(x)A^b_\nu(y)\Big] + \mathcal{O}(g^3). \end{aligned} \tag{15.11}$$

Compared with the Abelian case, we see that to order g^2, the non-Abelian result is given by making the replacement $e^2 \to g^2 C_2(r)$. Therefore we conclude that $V(R) = -g^2C_2(r)/4\pi R$ in non-Abelian case.

15.4 Scalar propagator in a gauge theory

In this problem we study very briefly the heat kernel representation of Green functions/propagator of a scalar field living within a gauge field background.

(a) To begin with, we consider the simplest case, in which the background gauge field vanishes. Then we can represent the Green function $D_F(x,y)$ of the Klein-Gordon equation, defined to be

$$(\partial^2+m^2)D_F(x,y)=-\mathrm{i}\delta^{(4)}(x-y) \tag{15.12}$$

with proper boundary conditions, by the following integral over the heat kernel function $D(x,y,T)$:

$$D_F(x,y)=\int_0^\infty \mathrm{d}T\,D(x,y,T). \tag{15.13}$$

The heat kernel satisfies the following "Schrödinger equation":

$$\Big[\mathrm{i}\frac{\partial}{\partial T}-(\partial^2+m^2)\Big]D(x,y,T)=\mathrm{i}\delta(T)\delta^{(4)}(x-y). \tag{15.14}$$

The solution to this equation can be represented by

$$\begin{aligned}
D(x,y,T)=\langle x|e^{-\mathrm{i}HT}|y\rangle &=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\langle x|k\rangle\langle k|e^{-\mathrm{i}HT}|k'\rangle\langle k'|y\rangle\\
&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4k'}{(2\pi)^4}\,e^{-\mathrm{i}(-k^2+m^2)T}e^{-\mathrm{i}k\cdot x+\mathrm{i}k'\cdot y}(2\pi)^4\delta^{(4)}(k-k')\\
&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{\mathrm{i}(k^2-m^2)T}e^{-\mathrm{i}k\cdot(x-y)},
\end{aligned} \tag{15.15}$$

with $H=\partial^2+m^2$. Integrating this result over T, with the $+\mathrm{i}\epsilon$ prescription, we recover the Feynman propagator for a scalar field:

$$\begin{aligned}
\int_0^\infty \mathrm{d}T\,D(x,y,T)&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-\mathrm{i}k\cdot(x-y)}\int_0^\infty\mathrm{d}T\,e^{\mathrm{i}(k^2-m^2+\mathrm{i}\epsilon)T}\\
&=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}e^{-\mathrm{i}k\cdot(x-y)}}{k^2-m^2+\mathrm{i}\epsilon}.
\end{aligned} \tag{15.16}$$

(b) Now let us turn on a background Abelian gauge field $A_\mu(x)$. The corresponding "Schrödinger equation" then becomes

$$\Big[\mathrm{i}\frac{\partial}{\partial T}-\Big(\big(\partial_\mu-\mathrm{i}eA_\mu(x)\big)^2+m^2\Big)\Big]D(x,y,T)=\mathrm{i}\delta(T)\delta^{(4)}(x-y), \tag{15.17}$$

the solution of which, $\langle x|e^{-\mathrm{i}HT}|y\rangle$, can also be expressed as a path integral,

$$\langle x|e^{-\mathrm{i}HT}|y\rangle=\lim_{N\to\infty}\int\prod_{i=1}^N\bigg(\mathrm{d}x_i\big\langle x_i\big|\exp\Big\{-\mathrm{i}\Delta t\big[\big(\partial_\mu-\mathrm{i}eA_\mu(x)\big)^2+m^2\big]\Big\}\big|x_{i-1}\big\rangle\bigg), \tag{15.18}$$

where we have identify $x=x_N$, $y=x_0$, and $\Delta t=T/N$. Then,

$$\begin{aligned}
&\langle x_i|e^{-\mathrm{i}\Delta t[(\partial_\mu-\mathrm{i}eA_\mu(x))^2+m^2]}|x_{i-1}\rangle\\
=&\int\frac{\mathrm{d}^4k_i}{(2\pi)^4}\langle x_i|e^{-\mathrm{i}\Delta t[\partial^2-\mathrm{i}eA_\mu(x)\partial^\mu+m^2]}|k_i\rangle\langle k_i|e^{-\mathrm{i}\Delta t[-\mathrm{i}e\partial^\mu A_\mu(x)-e^2A^2(x)]}|x_{i-1}\rangle
\end{aligned}$$

$$= \int \frac{d^4k_i}{(2\pi)^4} \langle x_i | e^{-i\Delta t[-k_i^2 + eA_\mu(x_i)k_i^\mu + m^2]} | k_i \rangle \langle k_i | e^{-i\Delta t[ek_i^\mu A_\mu(x_{i-1}) - e^2A^2(x_{i-1})]} | x_{i-1} \rangle$$
$$= \int \frac{d^4k_i}{(2\pi)^4} e^{-i\Delta t[-k_i^2 + ek_i \cdot (A(x_i) + A(x_{i-1})) - e^2A^2(x_{i-1}) + m^2 - i\epsilon]} e^{-ik_i \cdot (x_i - x_{i-1})}$$
$$= C \exp\Big[-\frac{i\Delta t}{4}\Big(\frac{x_i - x_{i-1}}{\Delta t} + eA(x_i) + eA(x_{i-1})\Big)^2 - i\Delta t(m^2 - e^2A^2(x_{i-1}))\Big]$$
$$\Rightarrow C \exp\Big[-\frac{i\Delta t}{4}\Big(\frac{dx}{dt}\Big)^2 - i\Delta t eA(x) \cdot \frac{dx}{dt} - i\Delta t m^2 \Big]. \tag{15.19}$$

In the last line we take the continuum limit, and C is an irrelevant normalization constant. Then we get

$$D(x, y, T) = \int \mathcal{D}x \, \exp\Big[-i \int_0^T dt \Big(\Big(\frac{dx}{dt}\Big)^2 + m^2 \Big) - ie \int_0^T dx(t) \cdot A(x(t)) \Big]. \tag{15.20}$$

15.5 Casimir operator computations

(a) In the language of angular momentum theory, we can take common eigenfunctions of $J^2 = \sum_a T^aT^a$ and $J_z = T^3$ to be the representation basis. Then the representation matrix for T^3 is diagonal:

$$t_j^3 = \text{diag}\,(-j, -j+1, \cdots, j-1, j).$$

Thus

$$\text{tr}\,(t_j^3 t_j^3) = \sum_{m=-j}^{j} m^2 = \tfrac{1}{3} j(j+1)(2j+1).$$

Then we have

$$\text{tr}\,(t_r^3 t_r^3) = \sum_i \text{tr}\,(t_{j_i}^3 t_{j_i}^3) = \tfrac{1}{3} \sum_i j_i(j_i+1)(2j_i+1) = C(r),$$

which implies that

$$3C(r) = \sum_i j_i(j_i+1)(2j_i+1). \tag{15.21}$$

(b) Let the $SU(2)$ subgroup be spanned by T^1, T^1 and T^3. Then in fundamental representation, the representation matrices for $SU(2)$ subgroup of $SU(N)$ can be taken as

$$t_N^i = \begin{pmatrix} \tau_i/2 & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.22}$$

Where τ_i $(i = 1, 2, 3)$ are Pauli matrices. We see that the representation matrices for $SU(2)$ decomposes into a doublet and $(N-2)$ singlet. Then it's easy to find that

$$C(N) = \tfrac{1}{3}\big(\tfrac{1}{2}(\tfrac{1}{2}+1)(2\cdot\tfrac{1}{2}+1)\big) = \tfrac{1}{2}, \tag{15.23}$$

by formula in (a).

In adjoint representation, the representation matrices $(t^i)_{ab} = \mathrm{i}f^{aib}$ $(a, b = 1, \cdots, N^2 - 1,\ i = 1, 2, 3)$. Thus we need to know some information about structure constants. Here we give a handwaving illustration by analyzing the structure of fundamental representation matrices a little bit more. Note that there're three types of representation matrices, listed as follows. For convenience, let's call them t_A, t_B and t_C:

$$t_A = \begin{pmatrix} A_{2\times 2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.24}$$

$$t_B = \begin{pmatrix} A_{2\times 2} & B_{2\times(N-2)} \\ B^\dagger_{(N-2)\times 2} & 0_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.25}$$

$$t_C = \begin{pmatrix} -\frac{1}{2}\,\mathrm{tr}\,(C) I_{2\times 2} & 0_{2\times(N-2)} \\ 0_{(N-2)\times 2} & C_{(N-2)\times(N-2)} \end{pmatrix}. \tag{15.26}$$

In which, t_A is just the representation matrices for $SU(2)$ subgroup. Thus we see that there are 3 t_A, $2(N-2)$ t_B and $(N-2)^2$ t_C in total. It's also obvious that there is no way to generate a t_A from commutators between two t_C or between a t_B and t_C, the only way to generate t_A are commutators between two t_A or between to t_B. Then, t_A commutators correspond to the triplet representation os $SU(2)$ subgroup, and $2(N-2)$-t_B commutators correspond to the doublet representation of $SU(2)$. In this way we see that adjoint representation matrices for $SU(2)$ subgroup decompose into 1 triplet, $2(N-1)$ doublets and $(N-2)^2$ singlets.

Then we can calculate $C(G)$, again, by using formula in (a), as:

$$C(G) = \tfrac{1}{3}\big[1(1+1)(2\cdot 1+1) + 2(N-2)\cdot\tfrac{1}{2}(\tfrac{1}{2}+1)(2\cdot\tfrac{1}{2}+1)\big] = N. \tag{15.27}$$

(c) Let $U \in SU(N)$ be $N\times N$ unitary matrix, S be a symmetric $N\times N$ matrix, and A be an antisymmetric $N\times N$ matrix. Then we can use S and A to build two representations for $SU(N)$ respectively, as

$$S \to USU^T, \qquad A \to UAU^T.$$

It's easy to verify that they are indeed representations. Let's denote these two representation by s and a. It's also obvious to see that the dimensions of s and a are $d(s) = N(N+1)/2$ and $d(a) = N(N-1)/2$ respectively.

Accordingly, the generator T^a acts on S and A as:

$$S \to T^a S + S(T^a)^T, \qquad A \to T^a A + A(T^a)^T. \tag{15.28}$$

To get $C_2(s)$ and $C_2(a)$, we can make use of the formula

$$d(r)C_2(r) = d(G)C(r). \tag{15.29}$$

Thus we need to calculate $C(r)$ and $C(a)$. By formula in (a), we can take an generator in $SU(2)$ subgroup to simplify the calculation. Let's take

$$t^3_N = \tfrac{1}{2}\mathrm{diag}(1, -1, 0, \cdots, 0),$$

Then we have:

$$S = \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \cdots & S_{nn} \end{pmatrix} \quad \rightarrow \quad t_N^3 S + S(t_N^3)^T = \frac{1}{2}\begin{pmatrix} 2S_{11} & 0 & S_{13} & \cdots & S_{1n} \\ 0 & 2S_{22} & S_{23} & \cdots & S_{2n} \\ S_{31} & S_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & 0 & \cdots & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & A_{12} & \cdots & A_{1n} \\ A_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{n-1,n} \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix} \quad \rightarrow \quad t_N^3 A + A(t_N^3)^T = \frac{1}{2}\begin{pmatrix} 0 & 0 & A_{13} & \cdots & A_{1n} \\ 0 & 0 & A_{23} & \cdots & A_{2n} \\ A_{31} & A_{32} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & 0 & \cdots & 0 \end{pmatrix}$$

Thus we see that the representation matrices for T^3, in both s representation and a representation, are diagonal. They are:

$$t_s^3 = \text{diag}(1, 0, \underbrace{\tfrac{1}{2} \cdots, \tfrac{1}{2}}_{N-2}, 1, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{N-2}, \underbrace{0, \cdots, 0}_{(N-2)(N-1)/2}); \tag{15.30}$$

$$t_a^3 = \text{diag}(0, \underbrace{\tfrac{1}{2}, \cdots, \tfrac{1}{2}}_{2(N-2)}, \underbrace{0, \cdots, 0}_{(N-2)(N-3)/2}). \tag{15.31}$$

Here we have rearrange the upper triangular elements of S and A by line.

Then we get

$$C(s) = \text{tr}\,(t_s^3)^2 = \tfrac{1}{2}(N+2); \tag{15.32}$$

$$C(a) = \text{tr}\,(t_a^3)^2 = \tfrac{1}{2}(N-2). \tag{15.33}$$

Then,

$$C_2(s) = \frac{d(G)C(s)}{d(s)} = \frac{(N^2-1)(N+2)/2}{N(N+1)/2} = \frac{(N-1)(N+2)}{N}; \tag{15.34}$$

$$C_2(a) = \frac{d(G)C(a)}{d(a)} = \frac{(N^2-1)(N-2)/2}{N(N-1)/2} = \frac{(N+1)(N-2)}{N}. \tag{15.35}$$

At last let's check the formula implied by (15.100) and (15.101):

$$\big(C_2(r_1) + C_2(r_2)\big)d(r_1)d(r_2) = \sum C_2(r_i)d(r_i), \tag{15.36}$$

in which the tensor product representation $r_1 \times r_2$ decomposes into a direct sum of irreducible representations r_i. In our case, the direct sum of representation s and a is equivalent to the tensor product representation of two copies of N. That is,

$$N \times N \cong s + a.$$

Thus, we have,

$$\big(C_2(N)+C_2(N)\big)d(N)d(N)=\Big[\frac{N^2-1}{2N}+\frac{N^2-1}{2N}\Big]N^2=N(N^2-1);$$

and

$$C_2(s)d(s)+C_2(a)d(a)=\big[C(s)+C(a)\big]d(G)=N(N^2-1).$$

Thus formula (15.36) indeed holds in our case.

Chapter 16

Quantization of Non-Abelian Gauge Theories

16.1 Arnowitt-Fickler gauge

In this problem we perform the Faddeev-Popov quantization of Yang-Mills theory in Arnowitt-Fickler gauge (also called axial gauge), namely $A^{3a} = 0$. More generally, we may write the gauge condition as $n_\mu A^{\mu a} = 0$ with n^μ an arbitrary space-like vector of unit norm ($n^2 = -1$). The condition $A^{3a} = 0$ corresponds simply to the choice $n^\mu = g^{\mu 3}$. This gauge has the advantage that the Faddeev-Popov ghosts do not propagate and do not couple to gauge fields, as we will show below.

Our starting point, the partition function, reads

$$Z = \int \mathcal{D}A_\mu \, \delta(n \cdot A^a) e^{\mathrm{i}S[A_\mu]} \det \Big(\frac{\delta_\alpha (n \cdot A^a)}{\partial \alpha^b} \Big), \tag{16.1}$$

with $S = -\frac{1}{4} \int \mathrm{d}^4 x \, (F^a_{\mu\nu})^2$ the classical action for the gauge field, and the Faddeev-Popov determinant is given by

$$\begin{aligned} \det \Big(\frac{\delta_\alpha (n \cdot A^a)}{\partial \alpha^b} \Big) &= \det \Big(\frac{1}{g} n_\mu \partial^\mu \delta^{ab} - f^{abc} n_\mu A^{\mu c} \Big) \\ &= \int \mathcal{D}b \mathcal{D}c \exp \Big[\mathrm{i} \int \mathrm{d}^4 x \, b^a \big(n_\mu \partial^\mu \delta^{ab} - f^{abc} n_\mu A^{\mu c} \big) c^b \Big]. \end{aligned} \tag{16.2}$$

When multiplied by the delta function $\delta(n \cdot A^a)$, the second term in the exponent above vanishes, which implies that the ghost and antighost do not interact with gauge field. Meanwhile, they do not propagator either, since there does not exist a canonical kinetic term for them. Therefore we can safely treat the Faddeev-Popov determinant as an overall normalization of the partition function and ignore it. Then, the partition function reduces to

$$\begin{aligned} Z &= \lim_{\xi \to 0} \int \mathcal{D}A_\mu \exp \Big[\mathrm{i} \int \mathrm{d}^4 x \Big(-\frac{1}{4} (F^a_{\mu\nu})^2 - \frac{1}{2\xi} (n \cdot A^a)^2 \Big) \Big] \\ &= \lim_{\xi \to 0} \int \mathcal{D}A_\mu \exp \Big[\mathrm{i} \int \mathrm{d}^4 x \Big(\tfrac{1}{2} A^a_\mu \big(g^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu - \tfrac{1}{\xi} n^\mu n^\nu \big) A^b_\nu \end{aligned}$$

$$
- g f^{abc}(\partial_\kappa A^a_\lambda)A^{\kappa b}A^{\lambda c} - \tfrac{1}{4}g^2 f^{eab}f^{ecd}A^a_\kappa A^b_\lambda A^{\kappa c}A^{\lambda d}\Big)\Big]. \tag{16.3}
$$

where we have convert the delta function $\delta(n\cdot A^a)$ into a limit of Gaussian function. Then we see that the three-point or four-point gauge boson vertices share the same Feynman rules with the ones in covariant gauge. The only difference arises from the propagator. Let us parameterize the propagator in momentum space as

$$
D^{\mu\nu}(k) = Ak^2 g^{\mu\nu} + Bk^\mu k^\nu + C(k^\mu n^\nu + n^\nu k^\mu) + Dn^\mu n^\nu. \tag{16.4}
$$

Then, the equation of motion satisfied by the propagator,

$$
\big(g^{\mu\nu}k^2 - k^\mu k^\nu - \tfrac{1}{\xi}n^\mu n^\nu\big)D_{\nu\lambda}(k) = g^\mu_\lambda \tag{16.5}
$$

gives

$$
A = -\frac{\mathrm{i}}{k^2}, \qquad B = -\frac{1+\xi k^2}{k\cdot n}C, \qquad C = -\frac{1}{k\cdot n}A, \qquad D = 0. \tag{16.6}
$$

Note that the gauge fixing parameter ξ should be sent to 0. Therefore the propagator reads

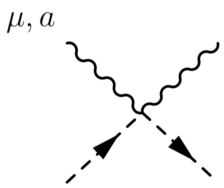
$$
D(k)^{\mu\nu} = -\frac{\mathrm{i}}{k^2}\bigg(g^{\mu\nu} + \frac{k^\mu k^\nu}{(k\cdot n)^2} - \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}\bigg). \tag{16.7}
$$

16.2 Scalar field with non-Abelian charge

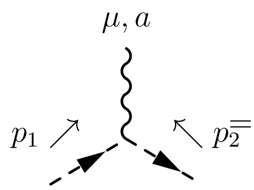
(a) Firstly we write down the Lagrangian for the Yang-Mills theory with charged scalar field, as

$$
\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + (\mathrm{D}_\mu\phi)^\dagger(\mathrm{D}^\mu\phi), \tag{16.8}
$$

where the covariant derivative $\mathrm{D}_\mu\phi = \big(\partial_\mu + \mathrm{i}gA^a_\mu t^a_r\big)\phi$ with t^a_r the matrices of gauge group generators in representation r. For simplicity we ignore the possible mass term for the scalar. Then, it is straightforward to derive the Feynman rules for this theory by expanding this Lagrangian. The rules for the propagator and self-interactions of gauge boson are independent of matter content and are the same with the ones given in Figure 16.1 in Peskin&Schroeder. The only new ingredients here are the gauge boson-scalar field interactions, which generate the following Feynman rules:

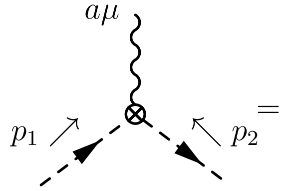


$$= \mathrm{i}g^2(t^a_r t^b_r + t^b_r t^a_r)g^{\mu\nu},$$

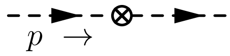


$$= -\mathrm{i}g t^a_r(p_1 - p_2)^\mu,$$

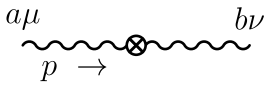
(b) To compute the β function of coupling g, we introduce some additional Feynman rules involving counterterms:



$$= \quad -\mathrm{i}g\delta_1 t_r^a (p_1 - p_2)^\mu,$$



$$= \quad \mathrm{i}p^2\delta_2 - \mathrm{i}\delta_m,$$



$$= \quad -\mathrm{i}\delta^{ab}\delta_3(p^2 g^{\mu\nu} - p^\mu p^\nu).$$

Then the β function is given by

$$\beta = gM\frac{\partial}{\partial M}\big(-\delta_1 + \delta_2 + \frac{1}{2}\delta_3\big). \tag{16.9}$$

To determine the counterterms, we evaluate the following relevant 1-loop diagrams. But the calculations can be simplified a lot if we observe that the combination $\delta_1 - \delta_2$ is determined by pure gauge sector, and is independent of matter content. This may be most easily seen from the counterterm relation $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$, where the right hand side comes from ghost contribution which is a pure gauge quantity. We will demonstrate this counterterm relation explicitly in the next problem for fermionic matter. Therefore, we can borrow directly the result of Peskin & Schroeder, or from the result of Problem 16.3(a),

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2}C_2(G)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.10}$$

On the other hand, δ_3 can be found by evaluating the loop corrections to the gauge boson's self-energy. The contributions from the gauge boson loop and ghost loop have already been given in eq.(16.71) in Peskin&Schroeder, while the rest of the contributions is from the scalar-loop, and is simply the result we have found in Problem 9.1(c), multiplied by the gauge factor $\mathrm{tr}\,(t_r^a t_r^b) = C(r)$ and the number of scalar n_s. Combining these two parts gives the divergent part of δ_3:

$$\delta_3 = \frac{g^2}{(4\pi)^2}\big[\frac{5}{3}C_2(G) - \frac{1}{3}n_s C(r)\big]\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.11}$$

Then it is straightforward to see that

$$\beta = -\frac{g^3}{(4\pi)^2}\Big(\frac{11}{3}C_2(G) - \frac{1}{3}n_s C(r)\Big). \tag{16.12}$$

16.3 Counterterm relations

In this problem we calculate the divergent parts of counterterms in Yang-Mills theory with Dirac spinors at 1-loop level, to verify the counterterm relations, which is a set of constraints set by gauge invariance. To begin with, let us rewrite the Lagrangian in its renormalized form, with counterterms separated, as

$$\begin{aligned}\mathcal{L} =& -\tfrac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \bar\psi(\mathrm{i}\partial\!\!\!/ - m)\psi - \bar c^a\partial^2 c^a \\ &+ gA_\mu^a\bar\psi\gamma^\mu t^a\psi - gf^{abc}(\partial_\mu A_\nu^a)A^{b\mu}A^{c\nu} \\ &- \tfrac{1}{4}g^2(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{c\mu}A^{d\nu}) - g\bar c^a f^{abc}\partial^\mu(A_\mu^b c^c) \\ &- \tfrac{1}{4}\delta_3(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \bar\psi(\mathrm{i}\delta_2\partial\!\!\!/ - \delta_m)\psi - \delta_2^c\bar c^a\partial^2 c^a \\ &+ g\delta_1 A_\mu^a\bar\psi\gamma^\mu\psi - g\delta_1^{3g}f^{abc}(\partial_\mu A_\nu^a)A_{b\mu}A^{c\nu} \\ &- \tfrac{1}{4}\delta_1^{4g}(f^{eab}A_\mu^a A_\nu^b)(f^{ecd}A^{c\mu}A^{d\nu}) - g\delta_1^c\bar c^a f^{abc}\partial^\mu(A_\mu^b c^c). \end{aligned} \tag{16.13}$$

Then the counterterm relations we will verify are

$$\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3 = \tfrac{1}{2}(\delta_1^{4g} - \delta_3) = \delta_1^c - \delta_2^c. \tag{16.14}$$

Note that δ_1 and δ_2 have been given in (16.84) and (16.77) in Peskin&Schroeder. Here we simply quote the results:

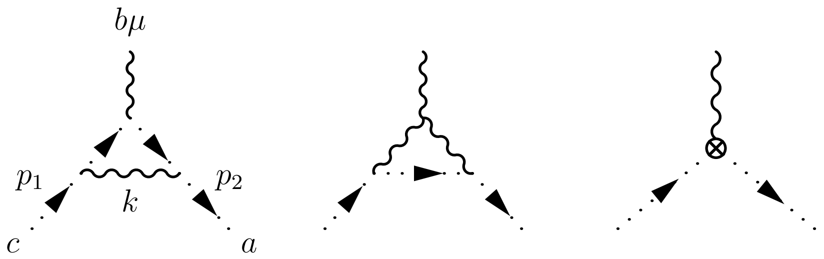
$$\delta_1 = -\frac{g^2}{(4\pi)^2}\big[C_2(r) + C_2(G)\big]\Big(\frac{2}{\epsilon} - \log M^2\Big), \tag{16.15}$$

$$\delta_2 = -\frac{g^2}{(4\pi)^2}C_2(r)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.16}$$

Therefore,

$$\delta_1 - \delta_2 = -\frac{g^2}{(4\pi)^2}C_2(G)\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.17}$$

(a) Firstly let us check the equality $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$. The 1-loop contributions to δ_1^c come from the following three diagrams:



The first diagram reads

$$\begin{aligned}&(-g)^3 f^{ade}f^{ebf}f^{fdc}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\frac{\mathrm{i}}{(p_1-k)^2}\frac{\mathrm{i}}{(p_2-k)^2}\cdot p_2^\nu(p_2-k)^\mu(p_1-k)_\nu \\ \Rightarrow &-g^3 f^{ade}f^{ebf}f^{fdc}\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{k^\mu(p_2\cdot k)}{k^6} = -\frac{1}{4}g^3 f^{ade}f^{ebf}f^{fdc}p_2^\mu\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^4} \\ \Rightarrow &-\frac{\mathrm{i}}{4}g^3 f^{ade}f^{ebf}f^{fdc}p_2^\mu\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}. \end{aligned} \tag{16.18}$$

The second diagram reads

$$
\begin{aligned}
&(-g)^2 g f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p_1-k)^2} \frac{-\mathrm{i}}{(p_2-k)^2} \\
&\quad \times p_{2\rho} k_\sigma \big[g^{\mu\rho}(k-p_2-q)^\sigma + g^{\mu\sigma}(q-p_1+k)^\rho + g^{\sigma\rho}(p_1+p_2-2k)^\mu \big] \\
\Rightarrow &-\mathrm{i}g^3 f^{ade} f^{ebf} f^{fdc} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^6} \big[p_2^\mu k^2 + k^\mu (k\cdot p_2) - 2k^\mu (k\cdot p_2) \big] \\
\Rightarrow &-\frac{3}{4}\mathrm{i}g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^4} \Rightarrow -\frac{3}{4}\mathrm{i}g^3 f^{ade} f^{ebf} f^{fdc} p_2^\mu \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned}
\tag{16.19}
$$

To simplify the structure constant product, we make use of the Jacobi identity,

$$
0 = f^{ebf}(f^{abd} f^{dce} + f^{bcd} f^{dae} + f^{cad} f^{dbe}) = 2 f^{abd} f^{dce} f^{ebf} - f^{caf} C_2(G),
$$

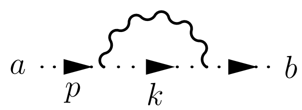
then we have

$$
f^{ade} f^{ebf} f^{fdc} = -\frac{1}{2} f^{abc} C_2(G). \tag{16.20}
$$

Note that the third diagram reads $-g\delta_1^c f^{abc} p_2^\mu$, thus we see that to make the sum of these three diagrams finite, the counterterm coefficient δ_1^c should be

$$
\delta_1^c \sim -\frac{g^2 C_2(G)}{2(4\pi)^2} \Big(\frac{2}{\epsilon} - \log M^2 \Big). \tag{16.21}
$$

Then consider δ_2^c. This coefficient should absorb the divergence from the following diagram:



This diagram reads

$$
\begin{aligned}
&(-g)^2 f^{bcd} f^{dca} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p-k)^2} (p\cdot k) \\
= &-g^2 C_2(G) \delta^{ab} \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{p\cdot(k'+xp)}{(k'^2-\Delta)^2} \\
\Rightarrow &-\frac{g^2}{2} C_2(G) \delta^{ab} p^2 \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon} + \text{terms indep. of } p^2.
\end{aligned}
\tag{16.22}
$$

The corresponding counterterm contributes $\mathrm{i}\delta_2^c p^2$, therefore we have

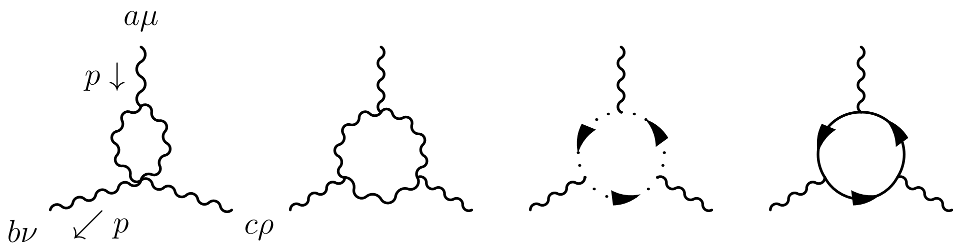
$$
\delta_2^c \sim \frac{g^2 C_2(G)}{2(4\pi)^2} \Big(\frac{2}{\epsilon} - \log M^2 \Big). \tag{16.23}
$$

Combining (16.21), (16.23) and (16.17), we see that the equality $\delta_1 - \delta_2 = \delta_1^c - \delta_2^c$ is satisfied.

(b) Now let's verify the equality $\delta_1 - \delta_2 = \delta_1^{3g} - \delta_3$. In this case the calculation turns out to be more cumbersome, though. The coefficient δ_3 has been given by (16.74) in Peskin&Schroeder. The result is

$$\delta_3 = \frac{g^2}{(4\pi)^2}\Big[\frac{5}{3}C_2(G) - \frac{4}{3}n_f C(r)\Big]\Big(\frac{2}{\epsilon} - \log M^2\Big). \tag{16.24}$$

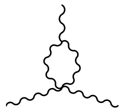
Thus we only need to calculate δ_1^{3g}. The relevant loop diagrams are listed as follows.



For simplicity, we have set the external momenta to be p, $-p$ and 0 for the three external gauge boson lines labeled with $(a\mu)$, $(b\nu)$ and $(c\rho)$. Then the contribution of the counterterm to this vertex is given by

$$g\delta_1^{3g} f^{abc}(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\rho\mu}p^\nu). \tag{16.25}$$

To extract the divergent part from δ_1^{3g}, we have to evaluate the loop diagrams shown above. Let us calculate them now in turn. The first diagram reads



$$\begin{aligned}
&= \tfrac{1}{2}g(-\mathrm{i}g^2)f^{ade}\big[f^{def}f^{bcf}(g^{\lambda\nu}g^{\kappa\rho} - g^{\lambda\rho}g^{\kappa\nu}) \\
&\quad + f^{dbf}f^{ecf}(g^{\lambda\kappa}g^{\nu\rho} - g^{\lambda\rho}g^{\kappa\nu}) + f^{dcf}f^{ebf}(g^{\lambda\kappa}g^{\nu\rho} - g^{\lambda\nu}g^{\kappa\rho})\big] \\
&\quad \times \int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{-\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(k-p)^2}\big[g^\mu_\lambda(p+k)_\kappa + g_{\lambda\kappa}(-2k+p)^\mu + g^\mu_\kappa(k-2p)_\lambda\big] \\
&= \tfrac{1}{2}\mathrm{i}g^3 f^{abc}C_2(G)\cdot\tfrac{3}{2}(g^{\lambda\nu}g^{\kappa\rho} - g^{\lambda\rho}g^{\kappa\nu}) \\
&\quad \times \int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2}\frac{1}{(k-p)^2}\big[g^\mu_\lambda(p+k)_\kappa + g_{\lambda\kappa}(-2k+p)^\mu + g^\mu_\kappa(k-2p)_\lambda\big] \\
&\Rightarrow \tfrac{1}{2}\mathrm{i}g^3 f^{abc}C_2(G)\cdot\tfrac{9}{2}(g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu)\int\frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^4} \\
&\Rightarrow \frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}.
\end{aligned} \tag{16.26}$$

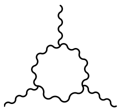
There are two additional diagrams associated with this diagram by the two cyclic permutations of the three external momenta. One gives

$$\frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon},$$

while the other yields zero. Therefore the sum of these three diagrams gives:

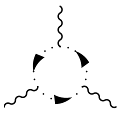
$$\frac{9}{4}\mathrm{i}g^3 f^{abc}C_2(G)(2g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu - g^{\nu\rho}p^\mu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}, \tag{16.27}$$

Then we come to the second diagram, which reads



$$
\begin{aligned}
&= g^3 f^{adf} f^{bed} f^{cfe} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{k^2} \frac{-\mathrm{i}}{(p+k)^2} \\
&\quad \times \big[g^{\mu\sigma}(2p+k)^\kappa + g^{\sigma\kappa}(-p-2k)^\mu + g^{\kappa\mu}(k-p)^\sigma \big] \\
&\quad \times \big[g^{\nu\lambda}(-p+k)_\sigma + g^\lambda_\sigma(-2k-p)^\nu + g^\nu_\sigma(2p+k)^\lambda \big] \\
&\quad \times \big[g^\rho_\kappa k_\lambda - 2g_{\kappa\lambda}k^\rho + g^\rho_\lambda k_\kappa \big] \\
&\Rightarrow \mathrm{i}g^3\big[-\tfrac{1}{2} f^{abc} C_2(G) \big] \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{2(1-x)k'^2}{(k'^2-\Delta)^3} \\
&\quad \times \frac{1}{4} \Big[2(8+15x) g^{\mu\nu} p^\rho + (30x-23)\big(g^{\mu\rho}p^\nu + g^{\nu\rho}p^\mu\big) \Big] \\
&\Rightarrow \mathrm{i}g^3\big[-\tfrac{1}{2} f^{abc} C_2(G) \big] \cdot \frac{13}{4} \big(2g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu - g^{\nu\rho}p^\mu \big) \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \frac{1}{k'^4} \\
&\Rightarrow -\frac{13}{8} \mathrm{i}g^3 f^{abc} C_2(G) \big(2g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu - g^{\nu\rho}p^\mu \big) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned} \tag{16.28}
$$

The third diagram reads



$$
\begin{aligned}
&= (-g)^3 f^{daf} f^{ebd} f^{fce} \int \frac{\mathrm{d}^d k}{(2\pi)^d} (-1) \Big(\frac{\mathrm{i}}{k^2} \Big)^2 \frac{\mathrm{i}}{(k+p)^2} \cdot (p+k)^\mu k^\nu k^\rho \\
&= -\mathrm{i}g^3 \cdot \tfrac{1}{2} f^{abc} C_2(G) \int \frac{\mathrm{d}^d k'}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{2(1-x)k'^2}{(k'^2-\Delta)^3} \\
&\quad \times \tfrac{1}{d} \big[-x g^{\mu\nu} p^\rho - x g^{\mu\rho} p^\nu + (1-x) g^{\nu\rho} p^\mu \big] \\
&\Rightarrow \frac{1}{24} \mathrm{i}g^3 f^{abc} C_2(G) (g^{\mu\nu}p^\rho + g^{\mu\rho}p^\nu - 2g^{\nu\rho}p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
\end{aligned} \tag{16.29}
$$

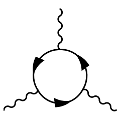
There is again a similar diagram with ghost loop running reversely, which gives

$$
\frac{1}{24} \mathrm{i}g^3 f^{abc} C_2(G) (g^{\mu\nu}p^\rho - 2g^{\mu\rho}p^\nu + g^{\nu\rho}p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
$$

Then these two diagrams with ghost loops sum to

$$
\frac{1}{24} \mathrm{i}g^3 f^{abc} C_2(G) (2g^{\mu\nu}p^\rho - g^{\mu\rho}p^\nu - g^{\nu\rho}p^\mu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}. \tag{16.30}
$$

Finally we consider the fourth diagram with fermion loop. There are also two copies with fermions running in opposite directions. One (shown in the figure) gives



$$
\begin{aligned}
&= n_f (\mathrm{i}g)^3 \, \mathrm{tr}\,(t^a t^c t^b) \int \frac{\mathrm{d}^d k}{(2\pi)^d} (-1) \, \mathrm{tr} \Big[\gamma^\mu \frac{\mathrm{i}}{\not{k}} \gamma^\rho \frac{\mathrm{i}}{\not{k}} \gamma^\nu \frac{\mathrm{i}}{\not{k} + \not{p}} \Big] \\
&\Rightarrow \frac{4}{3} n_f g^3 \, \mathrm{tr}\,(t^a t^c t^b) (2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon},
\end{aligned} \tag{16.31}
$$

while the other gives

$$
-\frac{4}{3} n_f g^3 \, \mathrm{tr}\,(t^a t^b t^c) (2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu) \cdot \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon}.
$$

Thus they sum to

$$\begin{aligned}
&\frac{4}{3}n_f g^3\,\mathrm{tr}\,\big(t^a[t^c,t^b]\big)(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}\\
=&-\frac{4}{3}\mathrm{i}n_f g^3 C(r)f^{abc}(2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu)\cdot\frac{\mathrm{i}}{(4\pi)^2}\frac{2}{\epsilon}.
\end{aligned}\tag{16.32}$$

Now, sum up the four groups of diagrams, we get

$$\frac{g^3}{(4\pi)^2}\frac{2}{\epsilon}f^{abc}\Big[\Big(-\frac{9}{4}+\frac{13}{8}-\frac{1}{24}\Big)C_2(G)+\frac{4}{3}n_f C(r)\Big](2g^{\mu\nu}p^\rho - g^{\nu\rho}p^\mu - g^{\mu\rho}p^\nu),\tag{16.33}$$

and consequently,

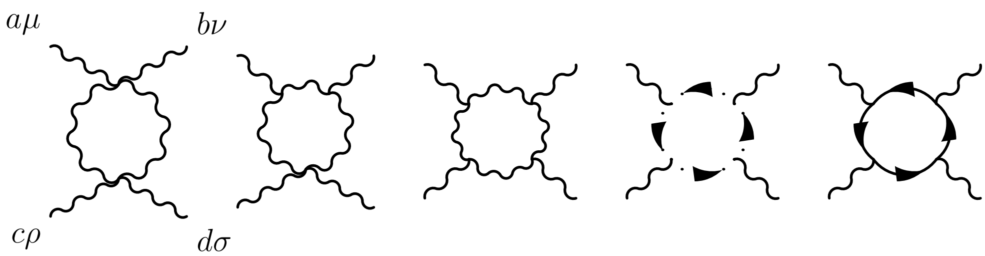
$$\delta_1^{3g} = \frac{g^2}{(4\pi)^2}\Big[\frac{2}{3}C_2(G)-\frac{4}{3}n_f C(r)\Big]\cdot\Big(\frac{2}{\epsilon}-\log M^2\Big).\tag{16.34}$$

Thus,

$$\delta_1^{3g}-\delta_3 = -\frac{g^2}{(4\pi)^2}C_2(G)\Big(\frac{2}{\epsilon}-\log M^2\Big),\tag{16.35}$$

which equals to $\delta_1-\delta_2$ (16.17), as expected.

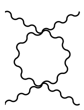
(c) Now let's move to the relation $\delta_1-\delta_2=\frac{1}{2}(\delta_1^{4g}-\delta_3)$. This time we have to evaluate δ_1^{4g}, which is determined by the divergent part of the following five types of diagrams:



Firstly the counterterm itself contributes to the 1-loop corrections with

$$\begin{aligned}
&-\delta_1^{4g}\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].
\end{aligned}\tag{16.36}$$

To evaluate the loop diagrams, we set all external momenta to zero for simplicity. The first diagram then reads



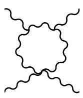
$$\begin{aligned}
&=\frac{1}{2}(-\mathrm{i}g^2)^2\Big[f^{abg}f^{efg}(g^{\mu\lambda}g^{\nu\kappa}-g^{\mu\kappa}g^{\nu\lambda})+f^{aeg}f^{bfg}(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\kappa}g^{\nu\lambda})\\
&\quad+f^{afg}f^{beg}(g^{\mu\nu}g^{\lambda\kappa}-g^{\mu\lambda}g^{\nu\kappa})\Big]\Big[f^{efh}f^{cdh}(g^\rho_\lambda g^\sigma_\kappa - g^\lambda_\sigma g^\kappa_\rho)\\
&\quad+f^{ech}f^{fdh}(g_{\lambda\kappa}g^{\rho\sigma}g^\lambda_\sigma g^\rho_\kappa)+f^{edh}f^{fch}(g_{\lambda\kappa}g^{\rho\sigma}-g^\rho_\lambda g^\sigma_\kappa)\Big]\int\frac{\mathrm{d}^d k}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2\\
\Rightarrow\ &\frac{\mathrm{i}g^4}{2(4\pi)^2}\frac{2}{\epsilon}\Big[f^{abg}f^{efg}f^{efh}f^{cdh}(2g^{\mu\rho}g^{\nu\sigma}-2g^{\mu\sigma}g^{\nu\rho})
\end{aligned}$$

$$
\begin{aligned}
&+ f^{abg} f^{efg} f^{ech} f^{fdh}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{abg} f^{efg} f^{edh} f^{fch}(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}) \\
&+ f^{aeg} f^{bfg} f^{efh} f^{cdh}(g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{afg} f^{beg} f^{efh} f^{cdh}(g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}) \\
&+ \operatorname{tr}(t^a t^b t^d t^c)(2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) + \operatorname{tr}(t^a t^b t^c t^d)(2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
&+ \operatorname{tr}(t^a t^b t^c t^d)(2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) + \operatorname{tr}(t^a t^b t^d t^c)(2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma})\Big] \\
= {} & \frac{\mathrm{i} g^4}{2(4\pi)^2} \frac{2}{\epsilon} \Big[\operatorname{tr}(t^a t^b t^c t^d)(4g^{\mu\nu} g^{\rho\sigma} - 8g^{\mu\rho} g^{\nu\sigma} + 10 g^{\mu\sigma} g^{\nu\rho}) \\
&+ \operatorname{tr}(t^a t^b t^d t^c)(4g^{\mu\nu} g^{\rho\sigma} + 10g^{\mu\rho} g^{\nu\sigma} - 8 g^{\mu\sigma} g^{\nu\rho})\Big],
\end{aligned} \tag{16.37}
$$

where we have used (16.20) and $f^{abc} = \mathrm{i}(t^a)_{bc}$ with t^a the generators in adjoint representation. There are two additional diagrams similar to this one, which can be obtained by exchange of labels as $(b\nu \leftrightarrow c\rho)$ and $(b\nu \leftrightarrow d\sigma)$. Therefore the total contribution from these three diagrams is

$$
\begin{aligned}
&\frac{\mathrm{i} g^4}{(4\pi)^2} \frac{2}{\epsilon} \Big[\operatorname{tr}(t^a t^b t^c t^d)(7g^{\mu\nu} g^{\rho\sigma} - 8g^{\mu\rho} g^{\nu\sigma} + 7 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \operatorname{tr}(t^a t^b t^d t^c)(7g^{\mu\nu} g^{\rho\sigma} + 7g^{\mu\rho} g^{\nu\sigma} - 8 g^{\mu\sigma} g^{\nu\rho}) \\
&\quad + \operatorname{tr}(t^a t^c t^b t^d)(-8g^{\mu\nu} g^{\rho\sigma} + 7g^{\mu\rho} g^{\nu\sigma} + 7 g^{\mu\sigma} g^{\nu\rho})\Big].
\end{aligned} \tag{16.38}
$$

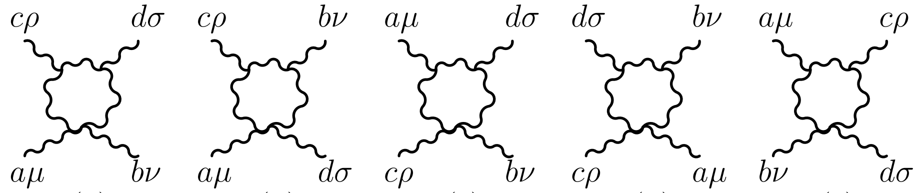
The second diagram has five additional counterparts. The one displayed in the figure reads



$$
\begin{aligned}
= {} & (-\mathrm{i}g^2) g^2 f^{aeg} f^{bgf} \big[f^{efh} f^{cdh} (g^\rho_\lambda g^\sigma_\kappa - g^\sigma_\lambda g^\rho_\kappa) \\
&+ f^{ech} f^{fdh} (g_{\lambda\kappa} g^{\rho\sigma} - g^\sigma_\lambda g^\kappa_\rho) + f^{edh} f^{fch} (g_{\lambda\kappa} g^{\rho\sigma} - g^\lambda_\rho g^\sigma_\kappa)\big] \\
&\times \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{-\mathrm{i}}{k^2}\right)^3 (g^{\mu\lambda} k^\tau - 2k^\mu + g^{\tau\mu} k^\lambda)(g^\nu_\tau k^\kappa - 2 g^\kappa_\tau k^\nu + g^{\kappa\nu} k_\tau) \\
= {} & g^4 f^{aeg} f^{bgf} \big[f^{efh} f^{cdh} (g^\rho_\lambda g^\sigma_\kappa - g^\sigma_\lambda g^\rho_\kappa) \\
&+ f^{ech} f^{fdh} (g_{\lambda\kappa} g^{\rho\sigma} - g^\sigma_\lambda g^\kappa_\rho) + f^{edh} f^{fch} (g_{\lambda\kappa} g^{\rho\sigma} - g^\lambda_\rho g^\sigma_\kappa)\big] \\
&\times \int \frac{\mathrm{d}^d k}{(2\pi)^d} \left(\frac{1}{k^2}\right)^2 \left(2 g^\mu_\lambda g^\nu_\kappa + 5 g^{\mu\nu} g_{\lambda\kappa} - 4 g^\mu_\kappa g^\nu_\lambda\right) \\
\Rightarrow {} & \frac{g^4}{4} \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon} \Big[f^{aeg} f^{bgf} f^{efh} f^{cdh} (6 g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&+ f^{aeg} f^{bgf} f^{ech} f^{fdh} (13 g^{\mu\nu} g^{\rho\sigma} + 4 g^{\mu\rho} g^{\nu\sigma} - 2 g^{\mu\sigma} g^{\nu\rho}) \\
&+ f^{aeg} f^{bgf} f^{edh} f^{fch} (13 g^{\mu\nu} g^{\rho\sigma} - 2 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho})\Big] \\
= {} & \frac{g^4}{4} \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon} \Big[\mathrm{i} f^{cdh} \operatorname{tr}(t^a t^b t^h)(6 g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
&- \operatorname{tr}(t^a t^b t^d t^c)(13 g^{\mu\nu} g^{\rho\sigma} + 4 g^{\mu\rho} g^{\nu\sigma} - 2 g^{\mu\sigma} g^{\nu\rho}) \\
&- \operatorname{tr}(t^a t^b t^c t^d)(13 g^{\mu\nu} g^{\rho\sigma} - 2 g^{\mu\rho} g^{\nu\sigma} + 4 g^{\mu\sigma} g^{\nu\rho})\Big] \\
= {} & -\frac{g^4}{4} \frac{\mathrm{i}}{(4\pi)^2} \frac{2}{\epsilon} \Big[\operatorname{tr}(t^a t^b t^c t^d)(13 g^{\mu\nu} g^{\rho\sigma} - 8 g^{\mu\rho} g^{\nu\sigma} + 10 g^{\mu\sigma} g^{\nu\rho})
\end{aligned}
$$

$$
+\ \mathrm{tr}\,(t^at^bt^dt^c)(13g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho})\Big], \tag{16.39}
$$

where we have used the fact that $f^{aeg}f^{bgf}f^{efh}f^{cdh}=\mathrm{i}f^{cdh}\,\mathrm{tr}\,(t^at^bt^h)=\mathrm{tr}\,\big(t^at^b[t^c,t^d]\big)$, and $f^{aeg}f^{bgf}f^{ech}f^{fdh}=-\,\mathrm{tr}\,(t^at^bt^dt^c)$, etc. There are additional five diagrams associated with this one, namely,



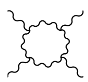
(a) (b) (c) (d) (e)

In the diagrams above, (a) gives the identical result to the one have just evaluated, while (b) and (c) give identical expression, so do (d) and (e). We can find (b) from the result above by the exchange ($a\mu\leftrightarrow c\rho$), and (d) by the exchange ($a\mu\leftrightarrow d\sigma$). Then we sum up all six diagrams, which is equivalent to summing the original one with (b) and (d) and multiplying the result by 2:

$$
\begin{aligned}
-\,2\cdot\frac{\mathrm{i}g^4}{4(4\pi)^2}\frac{2}{\epsilon}\Big[&\,\mathrm{tr}\,(t^at^bt^ct^d)(13g^{\mu\nu}g^{\rho\sigma}-8g^{\mu\rho}g^{\nu\sigma}+10g^{\mu\sigma}g^{\nu\rho})\\
&+\ \mathrm{tr}\,(t^at^bt^dt^c)(13g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho})\\
&+\ \mathrm{tr}\,(t^ct^bt^at^d)(13g^{\mu\sigma}g^{\nu\rho}-8g^{\mu\rho}g^{\nu\sigma}+10g^{\mu\nu}g^{\rho\sigma})\\
&+\ \mathrm{tr}\,(t^ct^bt^dt^a)(13g^{\mu\sigma}g^{\nu\rho}-8g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\rho}g^{\nu\sigma})\\
&+\ \mathrm{tr}\,(t^dt^bt^ct^a)(13g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\nu}g^{\rho\sigma}+10g^{\mu\sigma}g^{\nu\rho})\\
&+\ \mathrm{tr}\,(t^dt^bt^at^c)(13g^{\mu\rho}g^{\nu\sigma}-8g^{\mu\sigma}g^{\nu\rho}+10g^{\mu\nu}g^{\rho\sigma})\Big]\\
=-\,\frac{\mathrm{i}g^4}{2(4\pi)^2}\frac{2}{\epsilon}\Big[&\,\mathrm{tr}\,(t^at^bt^ct^d)(23g^{\mu\nu}g^{\rho\sigma}-16g^{\mu\rho}g^{\nu\sigma}+23g^{\mu\sigma}g^{\nu\rho})\\
&+\ \mathrm{tr}\,(t^at^bt^dt^c)(23g^{\mu\nu}g^{\rho\sigma}+23g^{\mu\rho}g^{\nu\sigma}-16g^{\mu\sigma}g^{\nu\rho})\\
&+\ \mathrm{tr}\,(t^at^ct^bt^d)(-16g^{\mu\nu}g^{\rho\sigma}+23g^{\mu\rho}g^{\nu\sigma}+23g^{\mu\sigma}g^{\nu\rho})\Big],
\end{aligned} \tag{16.40}
$$

where we use the cyclic symmetry of trace and also the relation $\mathrm{tr}\,(t^at^bt^ct^d)=\mathrm{tr}\,(t^dt^bt^ct^a)$.

The third diagram reads



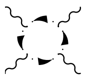
$$
\begin{aligned}
&= g^4f^{aeh}f^{bhg}f^{cfe}f^{dgf}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\Big(\frac{-\mathrm{i}}{k^2}\Big)^4(g^{\mu\lambda}k^\eta-2g^{\lambda\eta}k^\mu+g^{\eta\mu}k^\lambda)\\
&\quad\times(g^\nu_\eta k^\xi-2g^\xi_\eta k^\nu+g^{\xi\nu}k_\eta)(g^{\rho\kappa}k_\lambda-2g^\kappa_\lambda k^\rho+g^\rho_\lambda k^\kappa)\\
&\quad\times(g^\sigma_\xi k_\kappa-2g_{\xi\kappa}k^\sigma+g^\sigma_\kappa k_\xi)\\
&= g^4\,\mathrm{tr}\,(t^at^bt^dt^c)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2)^4}\big[34k^\mu k^\nu k^\rho k^\sigma+k^4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma})\\
&\quad+3k^2(g^{\mu\nu}k^\rho k^\sigma+g^{\mu\rho}k^\nu k^\sigma+g^{\nu\sigma}k^\mu k^\rho+g^{\rho\sigma}k^\mu k^\nu)\big]\\
&= g^4\,\mathrm{tr}\,(t^at^bt^dt^c)\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2)^2}\Big[\frac{34}{24}(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho})
\end{aligned}
$$

$$+ (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) + \frac{3}{2}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma})\Big]$$
$$= \frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t^a t^b t^d t^c)(47g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 17g^{\mu\sigma}g^{\nu\rho}). \tag{16.41}$$

Combined with the other two similar diagrams, we get

$$\frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t^a t^b t^c t^d)(47g^{\mu\nu}g^{\rho\sigma} + 17g^{\mu\rho}g^{\nu\sigma} + 47g^{\mu\sigma}g^{\nu\rho})$$
$$+\ \mathrm{tr}\,(t^a t^b t^d t^c)(47g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 17g^{\mu\sigma}g^{\nu\rho})$$
$$+\ \mathrm{tr}\,(t^a t^c t^b t^d)(17g^{\mu\nu}g^{\rho\sigma} + 47g^{\mu\rho}g^{\nu\sigma} + 47g^{\mu\sigma}g^{\nu\rho})\Big] \tag{16.42}$$

The fourth diagram with ghost loop is given by

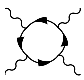


$$= (-g)^4 f^{eah} f^{hbg} f^{gdf} f^{fce} \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-1)\Big(\frac{\mathrm{i}}{k^2}\Big)^4 k^\mu k^\nu k^\rho k^\sigma$$
$$\Rightarrow -\frac{\mathrm{i}g^4}{24(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t^a t^b t^d t^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{16.43}$$

There are six distinct diagrams with ghost loops, with different permutations of external labels (Lorentz and gauge). They sum to

$$-\frac{\mathrm{i}g^4}{12(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t^a t^b t^c t^d) +\ \mathrm{tr}\,(t^a t^b t^d t^c) +\ \mathrm{tr}\,(t^a t^c t^b t^d)\Big]$$
$$\times (g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{16.44}$$

Finally the diagram with fermion loop reads



$$= (\mathrm{i}g)^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c) \int \frac{\mathrm{d}^d k}{(2\pi)^d}(-)\,\mathrm{tr}\Big[\gamma^\mu \frac{\mathrm{i}}{\not k}\gamma^\mu \frac{\mathrm{i}}{\not k}\gamma^\sigma \frac{\mathrm{i}}{\not k}\gamma^\rho \frac{\mathrm{i}}{\not k}\Big]$$
$$= -\,g^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c) \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2)^4}\big[4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})(k^2)^2$$
$$- 8(g^{\mu\nu}k^\rho k^\sigma + g^{\mu\rho}k^\nu k^\sigma + g^{\nu\sigma}k^\mu k^\rho + g^{\rho\sigma}k^\mu k^\nu)k^2 + 32k^\mu k^\nu k^\rho k^\sigma\big]$$
$$= -\,g^4 n_f\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c) \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{(k^2)^2}\big[4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$$
$$- 4(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) + \tfrac{4}{3}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})\big]$$
$$= -\frac{4\mathrm{i}g^4 n_f}{3(4\pi)^2}\frac{2}{\epsilon}\,\mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}). \tag{16.45}$$

Combined with the similar diagrams with different permutations, we get

$$-\frac{8\mathrm{i}g^4 n_f}{3(4\pi)^2}\frac{2}{\epsilon}\Big[\,\mathrm{tr}\,(t_r^a t_r^b t_r^c t_r^d)(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$
$$+\ \mathrm{tr}\,(t_r^a t_r^b t_r^d t_r^c)(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho})$$
$$+\ \mathrm{tr}\,(t_r^a t_r^c t_r^b t_r^d)(-2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}). \tag{16.46}$$

Now, we sum up the first four types of diagrams, namely, (16.38), (16.40), (16.42), and (16.44), and find the result to be

$$
\begin{aligned}
&\frac{2\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}\Big[\operatorname{tr}(t^at^bt^ct^d)(-g^{\mu\nu}g^{\rho\sigma}+2g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+\operatorname{tr}(t^at^bt^dt^c)(-g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}+2g^{\mu\sigma}g^{\nu\rho})\\
&\quad+\operatorname{tr}(t^at^ct^bt^d)(2g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\Big]\\
=&\frac{2\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}\Big[g^{\mu\nu}g^{\rho\sigma}\big(2\operatorname{tr}(t^at^ct^bt^d)-\operatorname{tr}(t^at^bt^ct^d)-\operatorname{tr}(t^at^bt^dt^c)\big)\\
&\quad+g^{\mu\rho}g^{\nu\sigma}\big(2\operatorname{tr}(t^at^bt^ct^d)-\operatorname{tr}(t^at^bt^dt^c)-\operatorname{tr}(t^at^ct^bt^d)\big)\\
&\quad+g^{\mu\sigma}g^{\nu\rho}\big(2\operatorname{tr}(t^at^bt^dt^c)-\operatorname{tr}(t^at^bt^ct^d)-\operatorname{tr}(t^at^ct^bt^d)\big)\Big]\\
=&\frac{\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}C_2(G)\big[g^{\mu\nu}g^{\rho\sigma}(-f^{ade}f^{bce}-f^{ace}f^{bde})\\
&\quad+g^{\mu\rho}g^{\nu\sigma}(f^{ade}f^{bce}-f^{abe}f^{cde})+g^{\mu\sigma}g^{\nu\rho}(f^{ace}f^{bde}+f^{abe}f^{cde})\big]\\
=&-\frac{\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}C_2(G)\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].
\end{aligned}
\tag{16.47}
$$

Similar manipulations on (16.46) gives

$$
\begin{aligned}
&-\frac{4\mathrm{i}g^4}{3(4\pi)^2}\frac{2}{\epsilon}n_fC_2(r)\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho})\\
&\quad+f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho})+f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma})\big].
\end{aligned}
\tag{16.48}
$$

Therefore, we finally find δ_1^{4g} to be

$$
-\frac{g^2}{3(4\pi)^2}\big[C_2(G)+4n_fC_2(r)\big]\Big(\frac{2}{\epsilon}-\log M^2\Big),
\tag{16.49}
$$

and it is straightforward to see that $\delta_1^{4g}-\delta_3=2(\delta_1-\delta_2)$.

Chapter 17

Quantum Chromodynamics

17.1 Two-Loop renormalization group relations

(a) In this problem we study the higher orders of QCD β function. Formally, we have

$$\beta(g) = -\frac{b_0}{(4\pi)^2}g^3 - \frac{b_1}{(4\pi)^4}g^5 - \frac{b_2}{(4\pi)^6}g^7 + \cdots . \tag{17.1}$$

The we can deduce the corresponding function for $\alpha_s \equiv g^2/(4\pi)$, namely,

$$\mu\frac{\partial\alpha_s}{\partial\mu} = -\frac{2b_0}{4\pi}\alpha_s^2 - \frac{2b_1}{(4\pi)^2}\alpha_s^3 - \frac{2b_2}{(4\pi)^3}\alpha_s^4 + \cdots . \tag{17.2}$$

Integrate this equation, we get

$$\int_\Lambda^Q \frac{\mathrm{d}\mu}{2\mu} = -\int_\infty^{\alpha_s(Q^2)} \mathrm{d}\alpha_s \left[\frac{b_0}{4\pi}\alpha_s^2 + \frac{b_1}{(4\pi)^2}\alpha_s^3 + \frac{b_2}{(4\pi)^3}\alpha_s^4 + \cdots\right]^{-1}. \tag{17.3}$$

The integral can be carried out approximately, as

$$\log(Q/\Lambda)^2 = \frac{4\pi}{b_0}\left[\frac{1}{\alpha_s(Q^2)} + \frac{b_1}{4\pi b_0}\log\frac{\alpha_s(Q^2)}{1+\frac{b_1}{4\pi b_0}\alpha_s(Q^2)} + \cdots\right]. \tag{17.4}$$

Then the running coupling $\alpha_s(Q^2)$ can be solved iteratively, to be,

$$\alpha_s(Q^2) = \frac{4\pi}{b_0}\left[\frac{1}{\log(Q/\Lambda)^2} - \frac{b_1}{b_0^2}\frac{\log\log(Q/\Lambda)^2}{[\log(Q/\Lambda)^2]^2} + \cdots\right]. \tag{17.5}$$

(b) Now we substitute (17.5) into the e^+e^- annihilation cross section, we get

$$\begin{aligned}
&\sigma(e^+e^- \to \text{hadrons})\\
&= \sigma_0 \cdot \Big(3\sum_f Q_f^2\Big) \cdot \Big[1 + \frac{\alpha_s}{\pi} + a_2\Big(\frac{\alpha_s}{\pi}\Big)^2 + \mathcal{O}(\alpha_s^3)\Big]\\
&= \sigma_0 \cdot \Big(3\sum_f Q_f^2\Big) \cdot \Big[1 + \frac{4}{b_0}\frac{1}{\log(Q/\Lambda)^2} - \frac{4b_1}{b_0^3}\frac{\log\log(Q/\Lambda)^2}{[\log(Q/\Lambda)^2]^2} + \cdots\Big].
\end{aligned} \tag{17.6}$$

Since the expression for the cross section is independent of renormalization scheme to the order showed above, we conclude that the β function coefficients b_0 and b_1 are also independent of the renormalization scheme.

17.2 A Direct test of the spin of the gluon

(a) We repeat the calculations in Part (c) of the Final Project I, with the gluon-quark vertex replaced by a Yukawa vertex.

$$\mathrm{i}\mathcal{M} = Q_q(-\mathrm{i}e)^2(-\mathrm{i}g)\bar{u}(k_1)\left[\frac{\mathrm{i}}{\not{k}_1+\not{k}_3}\gamma^\mu - \gamma^\mu\frac{\mathrm{i}}{\not{k}_2+\not{k}_3}\right]v(k_2)\frac{-\mathrm{i}}{q^2}\bar{v}(p_2)\gamma_\mu u(p_1) \tag{17.7}$$

Then, use the trick described in Final Project I, we have

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 =& \frac{Q_q^2g^2e^4}{4s^2}\,\mathrm{tr}\,(\gamma_\mu\not{p}_1\gamma_\rho\not{p}_2)\\ &\times\,\mathrm{tr}\left[\Big(\frac{1}{\not{k}_1+\not{k}_3}\gamma^\mu-\gamma^\mu\frac{1}{\not{k}_2+\not{k}_3}\Big)\not{k}_2\Big(\gamma^\rho\frac{1}{\not{k}_1+\not{k}_3}-\frac{1}{\not{k}_2+\not{k}_3}\gamma^\rho\Big)\not{k}_1\right]\\ =& \frac{32Q_q^2g^2e^4}{3s^2}(p_1\cdot p_2)(k_1\cdot k_3)(k_2\cdot k_3)\left[\frac{1}{(k_1+k_3)^2}+\frac{1}{(k_2+k_3)^2}\right]^2.\end{aligned} \tag{17.8}$$

Rewrite this in terms of x_q, $x_{\bar{q}}$ and x_3, we get

$$\begin{aligned}\frac{1}{4}\sum|\mathrm{i}\mathcal{M}|^2 =& \frac{4Q_q^2g^2e^4}{3s^2}(1-x_q)(1-x_{\bar{q}})\left[\frac{1}{1-x_q}+\frac{1}{1-x_{\bar{q}}}\right]^2\\ =& \frac{4Q_q^2g^2e^4}{3s^2}\frac{x_3^2}{(1-x_q)(1-x_{\bar{q}})}.\end{aligned} \tag{17.9}$$

Note the phase space integral for 3-body final state is deduced in Final Project 1 to be

$$\int \mathrm{d}\Pi_3 = \frac{s}{128\pi^3}\int \mathrm{d}x_q\mathrm{d}x_{\bar{q}},$$

thus the differential cross section is given by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_1\mathrm{d}x_2}(e^+e^-\to q\bar{q}S) = \frac{s}{128\pi^3}\cdot\tfrac{1}{4}|\mathcal{M}|^2 = \frac{4\pi\alpha^2Q_q^2}{3s}\cdot\frac{\alpha_g}{4\pi}\frac{x_3^2}{(1-x_q)(1-x_{\bar{q}})}. \tag{17.10}$$

(b) Now let $x_a > x_b > x_c$. Then there are six ways to associated the original three variables x_q, $x_{\bar{q}}$ and x_3 to these three ordered ones. Note that the integral measure $\mathrm{d}x_a\mathrm{d}x_b$ does not change for different possibilities since the change of integral variables $(x_q, x_{\bar{q}}) \to (x_q, x_3)$ or $\to (x_{\bar{q}}, x_3)$ generate an Jacobian whose absolute value is 1, due to the constraint $x_q + x_{\bar{q}} + x_3 = 2$. Therefore, summing up all 6 possibilities, we get

$$\begin{aligned}&\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_a\mathrm{d}x_b}(e^+e^-\to q\bar{q}S)\\ &\propto \frac{x_c^2}{(1-x_a)(1-x_b)}+\frac{x_b^2}{(1-x_c)(1-x_a)}+\frac{x_a^2}{(1-x_b)(1-x_c)},\end{aligned} \tag{17.11}$$

for $q\bar{q}S$ final state, and

$$\begin{aligned}&\frac{\mathrm{d}^2\sigma}{\mathrm{d}x_a\mathrm{d}x_b}(e^+e^-\to q\bar{q}S)\\ &\propto \frac{x_a^2+x_b^2}{(1-x_a)(1-x_b)}+\frac{x_b^2+x_c^2}{(1-x_b)(1-x_c)}+\frac{x_c^2+x_a^2}{(1-x_c)(1-x_a)},\end{aligned} \tag{17.12}$$

We plot these two distributions on the $x_a - x_b$ plain with the range $x_a > x_b > x_c$, as shown in Figure

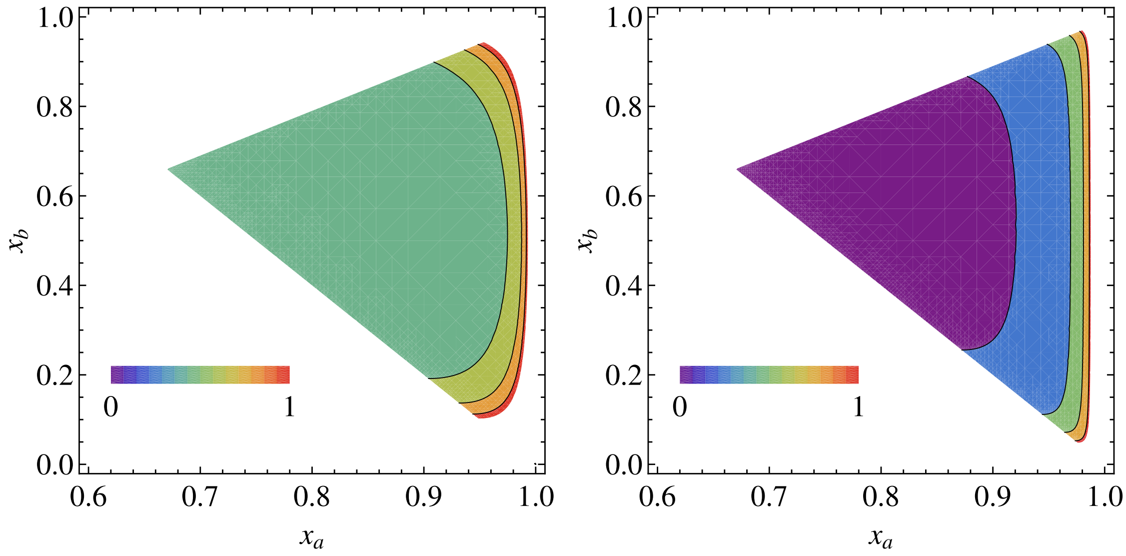


Figure 17.1: The differential cross sections of $e^+e^- \to q\bar{q}g$ as a function of x_1 and x_2, assuming gluon is a vector/scalar particle in left/right diagram.

17.3 Quark-gluon and gluon-gluon scattering

In this problem we evaluate the cross sections for two processes: (a) $q\bar{q} \to gg$, (b) $gg \to gg$.

(a) There are three diagrams contributing the process $q(k_1)\bar{q}(k_2) \to g(p_1)g(p_2)$ at the tree level, as shown in Fig. 17.11 of Peskin&Schroeder. The amplitudes associated with these diagrams are listed as follows:

$$\mathrm{i}\mathcal{M}_1 = (\mathrm{i}g)^2 \bar{v}(k_2)\not{\epsilon}^*(p_2)\frac{\mathrm{i}(\not{k}_1 - \not{p}_1)}{(k_1-p_1)^2}\not{\epsilon}^*(p_1)u(k_1)t^b t^a, \tag{17.13a}$$

$$\mathrm{i}\mathcal{M}_2 = (\mathrm{i}g)^2 \bar{v}(k_2)\not{\epsilon}^*(p_1)\frac{\mathrm{i}(\not{k}_1 - \not{p}_2)}{(k_1-p_2)^2}\not{\epsilon}^*(p_2)u(k_1)t^a t^b, \tag{17.13b}$$

$$\begin{aligned}\mathrm{i}\mathcal{M}_3 = {}& (\mathrm{i}g)gf^{abc}\big[g^{\mu\nu}(p_2-p_1)^\rho - g^{\nu\rho}(2p_2+p_1)^\mu + g^{\rho\mu}(p_2+2p_1)^\nu\big] \\ & \times \frac{-\mathrm{i}}{(k_1+k_2)^2}\bar{v}(k_2)\gamma_\rho u(k_1)\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)t^c.\end{aligned} \tag{17.13c}$$

It is convenient to evaluate these diagrams with initial and final states of definite helicities. By P and CP symmetry of QCD, there are only two independent processes, namely $q_L\bar{q}_R \to g_R g_R$ and $q_L\bar{q}_R \to g_R g_L$, that could be nonzero. Let's evaluate them in turn for the three diagrams. To begin with, we set up the kinematics:

$$\begin{aligned} k_1^\mu &= (E,0,0,E), & p_1^\mu &= (E, E\sin\theta, 0, E\cos\theta), \\ k_2^\mu &= (E,0,0,-E), & p_2^\mu &= (E, -E\sin\theta, 0, -E\cos\theta). \end{aligned} \tag{17.14}$$

Then,

$$u_L(k_1) = \sqrt{2E}(0,1,0,0), \qquad v_L(k_2) = \sqrt{2E}(1,0,0,0).$$

$$\begin{aligned} \epsilon^*_{L\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0, -\cos\theta, -\mathrm{i}, \sin\theta), & \epsilon^*_{R\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0, -\cos\theta, \mathrm{i}, \sin\theta), \\ \epsilon^*_{L\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0, \cos\theta, -\mathrm{i}, -\sin\theta), & \epsilon^*_{R\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0, \cos\theta, \mathrm{i}, -\sin\theta). \end{aligned} \tag{17.15}$$

Now we begin the calculation. (In the following we use $s_\theta \equiv \sin\theta$ and $c_\theta = \cos\theta$.)

$$\begin{aligned} \mathrm{i}\mathcal{M}_1(q_L\bar{q}_R \to g_R g_R) &= \frac{-\mathrm{i}g^2E^2t^bt^a}{t}(0,0,1,0)\begin{pmatrix} & & -s_\theta & 1+c_\theta \\ & & -1+c_\theta & s_\theta \\ s_\theta & -1-c_\theta & & \\ 1-c_\theta & -s_\theta & & \end{pmatrix} \\ &\times \begin{pmatrix} & & 1-c_\theta & -s_\theta \\ & & -s_\theta & -1+c_\theta \\ -1+c_\theta & s_\theta & & \\ s_\theta & 1-c_\theta & & \end{pmatrix}\begin{pmatrix} & & s_\theta & 1-c_\theta \\ & & -1-c_\theta & -s_\theta \\ -s_\theta & -1+c_\theta & & \\ 1+c_\theta & s_\theta & & \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ &= \mathrm{i}g^2t^bt^a\frac{2E^2}{t}(1-\cos\theta)\sin\theta = -\mathrm{i}g^2t^bt^a\sin\theta. \end{aligned} \tag{17.16}$$

$$\begin{aligned} \mathrm{i}\mathcal{M}_2(q_L\bar{q}_R \to g_R g_R) &= \frac{-\mathrm{i}g^2E^2t^at^b}{u}(0,0,1,0)\begin{pmatrix} & & s_\theta & 1-c_\theta \\ & & -1-c_\theta & -s_\theta \\ -s_\theta & -1+c_\theta & & \\ 1+c_\theta & s_\theta & & \end{pmatrix} \\ &\times \begin{pmatrix} & & 1+c_\theta & s_\theta \\ & & s_\theta & -1-c_\theta \\ -1-c_\theta & -s_\theta & & \\ -s_\theta & 1+c_\theta & & \end{pmatrix}\begin{pmatrix} & & -s_\theta & 1+c_\theta \\ & & -1+c_\theta & s_\theta \\ s_\theta & -1-c_\theta & & \\ 1-c_\theta & -s_\theta & & \end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ &= -\mathrm{i}g^2t^bt^a\frac{2E^2}{u}(1+\cos\theta)\sin\theta = \mathrm{i}g^2t^at^b\sin\theta. \end{aligned} \tag{17.17}$$

$$\begin{aligned} \mathrm{i}\mathcal{M}_3(q_L\bar{q}_R \to g_R g_R) &= \frac{g^2f^{abc}t^cE^2}{s}(0,0,1,0)\left[-4\begin{pmatrix} & & c_\theta & s_\theta \\ & & s_\theta & -c_\theta \\ -c_\theta & -s_\theta & & \\ -s_\theta & c_\theta & & \end{pmatrix}\right]\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\ &= g^2f^{abc}t^c\sin\theta = -\mathrm{i}g^2[t^a,t^b]\sin\theta. \end{aligned} \tag{17.18}$$

Thus we find that

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_Rg_R) = \big(\mathrm{i}\mathcal{M}_1 + \mathrm{i}\mathcal{M}_2 + \mathrm{i}\mathcal{M}_3\big)(q_L\bar{q}_R \to g_Rg_R) = 0. \tag{17.19}$$

In the same manner, we calculate the amplitude for $q_L\bar{q}_R \to g_Rg_L$. This time, we find:

$$\mathrm{i}\mathcal{M}_1(q_L\bar{q}_R \to g_Rg_L) = -\,\mathrm{i}g^2t^bt^a\sin\theta, \tag{17.20a}$$

$$\mathrm{i}\mathcal{M}_2(q_L\bar{q}_R \to g_Rg_L) = -\,\mathrm{i}g^2t^at^b\frac{t}{u}\sin\theta, \tag{17.20b}$$

$$\mathrm{i}\mathcal{M}_3(q_L\bar{q}_R \to g_R g_L) = 0, \tag{17.20c}$$

Therefore,

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_R g_L) = -\mathrm{i}g^2\Big(t^b t^a + t^a t^b \frac{t}{u}\Big)\sin\theta, \tag{17.21}$$

and by crossing symmetry,

$$\mathrm{i}\mathcal{M}(q_L\bar{q}_R \to g_L g_R) = -\mathrm{i}g^2\Big(t^b t^a + t^a t^b \frac{u}{t}\Big)\sin\theta. \tag{17.22}$$

There are two more nonzero amplitudes with $q_R\bar{q}_L$ initial states, which are identical to the amplitudes above. Then we find the spin- and color-summed/averaged squared amplitude to be

$$\begin{aligned}
&\frac{1}{3^2}\cdot\frac{1}{2^2}\sum_{\text{spin,color}}|\mathcal{M}|^2 \\
&= \frac{1}{36}\cdot 2\cdot g^4\sin^2\theta\bigg[\Big(\operatorname{tr}(t^b t^a t^a t^b) + 2\operatorname{tr}(t^b t^a t^b t^a)\frac{t}{u} + \operatorname{tr}(t^a t^b t^b t^a)\frac{t^2}{u^2}\Big) + (t\leftrightarrow u)\bigg] \\
&= \frac{8\pi^2\alpha_s^2}{9}(1-\cos^2\theta)\bigg[\Big(\frac{16}{3}\Big(1+\frac{t^2}{u^2}\Big) - \frac{4t}{3u}\Big) + (t\leftrightarrow u)\bigg] \\
&= \frac{512\pi^2\alpha_s^2}{27}\bigg[\frac{t}{u}+\frac{u}{t} - \frac{9(t^2+u^2)}{4s^2}\bigg].
\end{aligned} \tag{17.23}$$

Therefore the differential cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{32\pi\alpha_s^2}{27s^2}\bigg[\frac{t}{u}+\frac{u}{t} - \frac{9(t^2+u^2)}{4s^2}\bigg]. \tag{17.24}$$

(b) Now consider the process $g(k_1)g(k_2)\to g(p_1)g(p_2)$. The four tree level diagrams are shown in Fig. 17.12 of Peskin&Schroeder. Their amplitudes are given by:

$$\begin{aligned}
\mathrm{i}\mathcal{M}_1 = {}& g^2 f^{abc}f^{cde}\frac{-\mathrm{i}}{s}\big[g^{\mu\nu}(k_1-k_2)^\lambda + g^{\nu\lambda}(k_1+2k_2)^\mu - g^{\lambda\mu}(2k_1+k_2)^\nu\big] \\
&\times\big[g^{\rho\sigma}(p_2-p_1)_\lambda - g^\sigma_\lambda(p_1+2p_2)^\rho + g^\rho_\lambda(p_1+2p_2)^\rho\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25a}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_2 = {}& g^2 f^{ace}f^{bde}\frac{-\mathrm{i}}{t}\big[g^{\mu\rho}(k_1+p_1)^\lambda - g^{\rho\lambda}(2p_1-k_1)^\mu - g^{\lambda\nu}(2k_1-p_1)^\rho\big] \\
&\times\big[g^{\nu\sigma}(k_2+p_2)_\lambda - g^\sigma_\lambda(2p_2-k_2)^\nu + g^\nu_\lambda(p_2-2k_2)^\sigma\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25b}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_3 = {}& g^2 f^{ade}f^{bce}\frac{-\mathrm{i}}{u}\big[g^{\mu\sigma}(k_1+p_2)^\lambda - g^{\sigma\lambda}(2p_2-k_1)^\mu - g^{\lambda\nu}(2k_1-p_2)^\sigma\big] \\
&\times\big[g^{\nu\rho}(k_2+p_1)_\lambda - g^\rho_\lambda(2p_1-k_2)^\nu + g^\nu_\lambda(p_1-2k_2)^\rho\big]\epsilon_\mu(k_1)\epsilon_\nu(k_2)\epsilon^*_\rho(p_1)\epsilon^*_\sigma(p_2),
\end{aligned} \tag{17.25c}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}_4 = {}& -\mathrm{i}g^2\big[f^{abe}f^{cde}\big(\epsilon(k_1)\cdot\epsilon^*(p_1)\epsilon(k_2)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_2)\epsilon(k_2)\cdot\epsilon^*(p_1)\big) \\
&+ f^{ace}f^{bde}\big(\epsilon(k_1)\cdot\epsilon(k_2)\epsilon^*(k_1)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_2)\epsilon(k_2)\cdot\epsilon^*(p_1)\big) \\
&+ f^{ade}f^{bce}\big(\epsilon(k_1)\cdot\epsilon(k_2)\epsilon^*(p_1)\cdot\epsilon^*(p_2) - \epsilon(k_1)\cdot\epsilon^*(p_1)\epsilon(k_2)\cdot\epsilon^*(p_2)\big)\big].
\end{aligned} \tag{17.25d}$$

The choice for all external momenta and final states polarizations are the same with that in (a). Now to evaluate the amplitude $g_R g_R \to g_R g_R$, we also need the initial states polarization vectors for right-handed gluons with momenta k_1 and k_2, which are given by

$$\epsilon_R^\mu(k_1) = \tfrac{1}{\sqrt{2}}(0,1,\mathrm{i},0), \qquad \epsilon_R^\mu(k_2) = \tfrac{1}{\sqrt{2}}(0,-1,\imath,0). \tag{17.26}$$

Then after some calculations, we find,

$$\begin{aligned}
\mathrm{i}\mathcal{M}_1 =& -\mathrm{i}g^2 f^{abe} f^{cde}\cos\theta, && (17.27\text{a})\\
\mathrm{i}\mathcal{M}_2 =& \,\mathrm{i}g^2 f^{ace} f^{bde}\frac{19+7\cos\theta-11\cos^2\theta+\cos^3\theta}{4(1-\cos\theta)}; && (17.27\text{b})\\
\mathrm{i}\mathcal{M}_3 =& \,\mathrm{i}g^2 f^{ade} f^{bce}\frac{19-7\cos\theta-11\cos^2\theta-\cos^3\theta}{4(1+\cos\theta)}; && (17.27\text{c})\\
\mathrm{i}\mathcal{M}_4 =& -\mathrm{i}g^2\Big[f^{abe}f^{cde}\cos\theta+\tfrac{1}{4}f^{ace}f^{bde}(3+2\cos\theta-\cos^2\theta)\\
&+\tfrac{1}{4}f^{ade}f^{bce}(3-2\cos\theta-\cos^2)\Big]. && (17.27\text{d})
\end{aligned}$$

The sum of these four amplitudes is

$$\begin{aligned}
\mathrm{i}\mathcal{M}(g_Rg_R\to g_Rg_R) =& -2\mathrm{i}g^2\Big[f^{abe}f^{cde}\cos\theta - f^{ace}f^{bde}\Big(\frac{2}{1-\cos\theta}+\cos\theta\Big)\\
&- f^{ade}f^{bce}\Big(\frac{2}{1+\cos\theta}-\cos\theta\Big)\Big]\\
=&\, 4\mathrm{i}g^2\Big[f^{ace}f^{bde}\frac{1}{1-\cos\theta}+f^{ade}f^{bce}\frac{1}{1+\cos\theta}\Big]\\
=& -2\mathrm{i}g^2\Big[f^{ace}f^{bde}\frac{s}{t}+f^{ade}f^{bce}\frac{s}{u}\Big].
\end{aligned} \tag{17.28}$$

We can also obtain the amplitudes for $g_Lg_R\to g_Lg_R$ and $g_Lg_R\to g_Rg_L$ from the result above by crossing symmetry, namely the change of variables $(s,b)\leftrightarrow(u,d)$ and $(s,b)\leftrightarrow(t,c)$, which gives

$$\mathrm{i}\mathcal{M}(g_Lg_R\to g_Lg_R) = 2\mathrm{i}g^2\Big[f^{ace}f^{bde}\frac{u}{t}+f^{abe}f^{cde}\frac{u}{s}\Big], \tag{17.29}$$

$$\mathrm{i}\mathcal{M}(g_Lg_R\to g_Rg_L) = -2\mathrm{i}g^2\Big[f^{abe}f^{cde}\frac{t}{s}-f^{ade}f^{bce}\frac{t}{u}\Big]. \tag{17.30}$$

The amplitudes for $g_Lg_L\to g_Lg_L$, $g_Rg_L\to g_Rg_L$ and $g_Rg_L\to g_Lg_R$ are identical to the amplitudes for $g_Rg_R\to g_Rg_R$, $g_Lg_R\to g_Lg_R$ and $g_Lg_R\to g_Rg_L$, respectively, due to parity conservation of QCD. It can be shown by the conservation of angular momentum that other helicity amplitudes all vanish. Therefore we have found all required amplitude. To get the cross section, we take the square of these results.

$$\begin{aligned}
&\sum|\mathcal{M}(g_Rg_R\to g_Rg_R)|^2\\
&=4g^4\Big[f^{ace}f^{bde}f^{acf}f^{bdf}\frac{s^2}{t^2}+f^{ade}f^{bce}f^{adf}f^{bcf}\frac{s^2}{u^2}+2f^{ace}f^{bde}f^{adf}f^{bcf}\frac{s^2}{tu}\Big]\\
&=4g^4\Big[\operatorname{tr}(t^at^at^bt^b)\Big(\frac{s^2}{t^2}+\frac{s^2}{u^2}\Big)+2\operatorname{tr}(t^at^bt^at^b)\frac{s^2}{tu}\Big]
\end{aligned}$$

$$= 288g^4\Big(\frac{s^2}{t^2}+\frac{s^2}{u^2}+\frac{s^2}{tu}\Big), \tag{17.31}$$

where t^a is the generator of $SU(3)$ group in adjoint representation which is related to the structure constants by $f^{abc} = \mathrm{i}(t^a)_{bc}$. Thus $\mathrm{tr}\,(t^at^at^bt^b) = (C_2(G))^2 d(G) = 72$, and

$$\begin{aligned}\mathrm{tr}\,(t^at^bt^at^b) = {} & \mathrm{tr}\,(t^at^b[t^a,t^b]) + \mathrm{tr}\,(t^at^at^bt^b) = \tfrac{1}{2}\mathrm{i}f^{abc}\,\mathrm{tr}\,([t^a,t^b]t^c) + (C_2(G))^2 d(G)\\ = {} & -\tfrac{1}{2}f^{abc}f^{abd}\,\mathrm{tr}\,(t^ct^d) + (C_2(G))^2 d(G) = \tfrac{1}{2}(C_2(G))^2 d(G),\end{aligned}$$

which is 36 for $SU(3)$. Similarly, we can work out the square of other amplitudes, to be

$$\sum|\mathcal{M}(g_Lg_R\to g_Lg_R)|^2 = 288g^4\Big(\frac{u^2}{t^2}+\frac{u^2}{s^2}+\frac{u^2}{st}\Big), \tag{17.32}$$

$$\sum|\mathcal{M}(g_Lg_R\to g_Rg_L)|^2 = 288g^4\Big(\frac{t^2}{s^2}+\frac{t^2}{u^2}+\frac{t^2}{su}\Big). \tag{17.33}$$

Therefore, the spin-averaged and squared amplitudes is

$$\begin{aligned}\frac{1}{8^2}\cdot\frac{1}{2^2}\sum|\mathcal{M}^2| &= \frac{1}{8^2\cdot 2^2}\cdot 2\cdot 288g^4\Big(6-\frac{2tu}{s^2}-\frac{2us}{t^2}-\frac{2st}{u^2}\Big)\\ &= 72\pi^2\alpha_s^2\Big(3-\frac{tu}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\Big).\end{aligned} \tag{17.34}$$

Thus the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}(gg\to gg) = \frac{9\pi\alpha_s^2}{2s^2}\Big(3-\frac{tu}{s^2}-\frac{us}{t^2}-\frac{st}{u^2}\Big). \tag{17.35}$$

17.4 The gluon splitting function

In this problem we calculate the gluon splitting function $P_{g\leftarrow g}(z)$ by evaluating the amplitude of the virtual process $g\to gg$, as shown in Fig. 17.2.

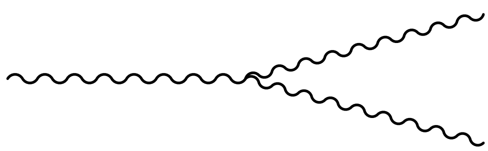


Figure 17.2: The Gluon splitting process.

The momenta of initial and final states are taken to be the same with that of Fig. 17.16 of Peskin&Schroeder. That is, we have

$$p = (p,0,0,p), \qquad q = (zp, p_\perp, 0, zp), \qquad k = ((1-z)p, -p_\perp, 0, (1-z)p), \tag{17.36}$$

and the polarization vectors associated with gluons are,

$$\epsilon_L^i(p) = \tfrac{1}{\sqrt{2}}(1,-\mathrm{i},0), \qquad \epsilon_R^i(p) = \tfrac{1}{\sqrt{2}}(1,\mathrm{i},0),$$

$$\epsilon_L^i(q) = \tfrac{1}{\sqrt{2}}(1,-\mathrm{i},-\frac{p_\perp}{zp}), \qquad \epsilon_R^i(q) = \tfrac{1}{\sqrt{2}}(1,\mathrm{i},-\frac{p_\perp}{zp}),$$

$$\epsilon_L^i(k) = \tfrac{1}{\sqrt{2}}(1, -\mathrm{i}, \frac{p_\perp}{(1-z)p}), \qquad \epsilon_R^i(k) = \tfrac{1}{\sqrt{2}}(1, \mathrm{i}, \frac{p_\perp}{(1-z)p}). \tag{17.37}$$

Then we can evaluate the amplitude for the process $g \to gg$ directly, which is given by

$$\begin{aligned}\mathrm{i}\mathcal{M}^{abc} = gf^{abc}\big[&\big(\epsilon^*(q)\cdot\epsilon(p)\big)\big((p+q)\cdot\epsilon^*(k)\big) + \big(\epsilon^*(q)\cdot\epsilon^*(k)\big)\big((k-q)\cdot\epsilon(p)\big)\\ &- \big(\epsilon^*(k)\cdot\epsilon(p)\big)\big((p+k)\cdot\epsilon^*(q)\big)\big].\end{aligned} \tag{17.38}$$

We evaluate the amplitudes with definite initial and final polarizations in turn:

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p)\to g_L(q)g_L(k)) = \sqrt{2}\Big(\frac{1}{1-z}+\frac{1}{z}\Big)gf^{abc}p_\perp, \tag{17.39a}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p)\to g_L(q)g_R(k)) = \frac{\sqrt{2}z}{1-z}gf^{abc}p_\perp, \tag{17.39b}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p)\to g_R(q)g_L(k)) = \frac{\sqrt{2}(1-z)}{z}gf^{abc}p_\perp, \tag{17.39c}$$

$$\mathrm{i}\mathcal{M}^{abc}(g_L(p)\to g_R(q)g_R(k)) = 0. \tag{17.39d}$$

By parity invariance, the amplitudes with right-handed initial gluon are dictated by the results above. Note further that $f^{abc}f^{abc} = 24$, thus we have

$$\begin{aligned}\tfrac{1}{2}\cdot\tfrac{1}{8}\sum_{\text{spin,color}}|\mathcal{M}|^2 &= \tfrac{1}{2}\cdot\tfrac{1}{8}\cdot 2\cdot 24\cdot 2g^2p_\perp^2\\ &\quad\times\Big[\Big(\frac{1}{1-z}+\frac{1}{z}\Big)^2 + \frac{z^2}{(1-z)^2} + \frac{(1-z)^2}{z^2}\Big]\\ &= \frac{12g^2p_\perp^2}{z(1-z)}\Big[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\Big]\\ &= \frac{2e^2p_\perp^2}{z(1-z)}\cdot P^{(0)}_{g\leftarrow g}(z),\end{aligned} \tag{17.40}$$

where the superscript (1) represents the part of the splitting function contributed from the diagram calculated above, in parallel with the notation of Peskin&Schroeder. (See 17.100. for instance.) Therefore we get

$$P^{(1)}_{g\leftarrow g} = 6\Big[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\Big]. \tag{17.41}$$

Besides, there should be a term proportional to $\delta(1-z)$ in $P_{g\leftarrow g}$, which comes from the zeroth order, as well as the corrections from $P_{q\leftarrow g}$ and $P_{g\leftarrow g}$, where $P_{q\leftarrow g}(z) = \frac{1}{2}(z^2+(1-z)^2)$. Now let's take it to be $A\delta(1-z)$, then the coefficient A can be determined by the following normalization condition (namely the momentum conservation):

$$1 = \int_0^1 \mathrm{d}z\, z\big[2n_f P_{q\leftarrow g}(z) + P^{(1)}_{g\leftarrow g}(z) + A\delta(1-z)\big], \tag{17.42}$$

where n_f is the number of fermion types, and the coefficient 2 is from contributions of both quarks and anti-quarks. To carry out the integral, we use the prescription $\frac{1}{1-z} \to \frac{1}{(1-z)_+}$, then it is straightforward to find that $A = \frac{11}{2} - \frac{1}{3}n_f$. Therefore,

$$P_{g\leftarrow g} = 6\Big[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\Big] + \Big(\frac{11}{2} - \frac{n_f}{3}\Big)\delta(1-z). \tag{17.43}$$

17.5 Photoproduction of heavy quarks

In this problem we study the production of a pair of heavy quark-antiquark by the scattering of a photon off a proton. At the leading order at the parton level, the process is contributed from the photon-gluon scattering, as shown in Figure 17.3.

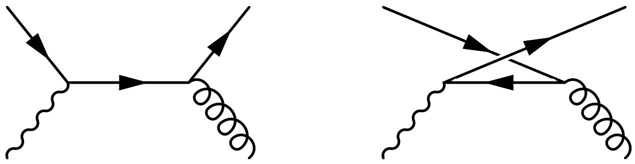


Figure 17.3: Tree diagrams for the photoproduction of heavy quarks at the parton level.

The corresponding amplitude can be read from a similar process $\gamma\gamma \to e^+e^-$ in QED. From (5.105) of Peskin & Schroeder, we have the amplitude for $e^+e^- \to 2\gamma$, which reads (adapted to our notation for external momenta)

$$\frac{1}{4}\sum|\mathcal{M}(e^+e^- \to 2\gamma)|^2 = 2e^4\Big[\frac{k_1\cdot p_2}{k_1\cdot p_1} + \frac{k_1\cdot p_1}{k_1\cdot p_2} + 2m^2\Big(\frac{1}{k_1\cdot p_1} + \frac{1}{k_1\cdot p_2}\Big) - m^4\Big(\frac{1}{k_1\cdot p_1} + \frac{1}{k_1\cdot p_2}\Big)^2\Big]. \tag{17.44}$$

Then the amplitude $\mathcal{M}(\gamma g \to Q\bar{Q})$ can be obtained by making the exchange $(k_1, k_2) \leftrightarrow (p_1, p2)$, replacing e^4 by e^2g^2, and also including the factor $\frac{1}{8}Q_q^2\,\mathrm{tr}\,(t^at^a) = \frac{1}{2}Q_q^2$ taking account of the color average, the electric charge of quarks, and the summation of color indices, respectively. Then the amplitude in the present case is

$$\frac{1}{4\cdot 8}\sum|\mathcal{M}(\gamma g \to Q\bar{Q})|^2 = e^2g^2Q_q^2\Big[\frac{p_1\cdot k_2}{p_1\cdot k_1} + \frac{p_1\cdot k_1}{p_1\cdot k_2} + 2m^2\Big(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\Big) - m^4\Big(\frac{1}{p_1\cdot k_1} + \frac{1}{p_1\cdot k_2}\Big)^2\Big]. \tag{17.45}$$

In parton's center-of-mass frame, we have $k_1 = (E,0,0,E)$, $k_2 = (E,0,0,-E)$, $p_1 = (E, p\sin\theta, 0, p\cos\theta)$ and $p_2 = (E, -p\sin\theta, 0, -p\cos\theta)$, with $p^2 = E^2 - m^2$. Then $p_1\cdot k_1 = E(E - p\cos\theta)$ and $p_1\cdot k_2 = E(E + p\cos\theta)$. Then the differential cross section is

$$\frac{\mathrm{d}\hat{\sigma}}{\mathrm{d}\cos\theta} = \frac{\pi\alpha\alpha_sQ_q^2}{16}\frac{p}{E^3}\Big[\frac{E^2 + p^2\cos^2\theta - 2m^2}{E^2 - p^2\cos^2\theta} - \frac{2m^4}{(E^2 - p^2\cos^2\theta)^2}\Big]. \tag{17.46}$$

Then the cross section for photon and proton initial state is given by

$$\sigma(\gamma(k_1) + p(k_2) \to Q\bar{Q}) = \int \mathrm{d}x\, f_g(x)\hat{\sigma}(\gamma(k_1) + g(xk_2) \to Q\bar{Q}). \tag{17.47}$$

17.6 Behavior of parton distribution functions at small x

(a) In this problem we study the solution of A-P equations at small x with certain approximations. Firstly, we show that the A-P equations,

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{g\leftarrow q}(z)\sum_f\left(f_f\left(\frac{x}{z},Q\right)+f_{\bar f}\left(\frac{x}{z},Q\right)\right)\right.$$
$$\left.+\,P_{g\leftarrow g}(z)f_g\left(\frac{x}{z},Q\right)\right], \tag{17.48}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z)f_f\left(\frac{x}{z},Q\right)+P_{q\leftarrow g}(z)f_g\left(\frac{x}{z},Q\right)\right], \tag{17.49}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_{\bar f}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z)f_{\bar f}\left(\frac{x}{z},Q\right)+P_{q\leftarrow g}(z)f_g\left(\frac{x}{z},Q\right)\right], \tag{17.50}$$

can be rewritten as a differential equation with variable $\xi = \log\log(Q^2/\Lambda^2)$. To see this, we note that $\mathrm{d}/\mathrm{d}\log Q = 2e^{-\xi}\mathrm{d}/\mathrm{d}\xi$, and to 1-loop order, $\alpha_s(Q) = 2\pi/\big(b_0\log(Q/\Lambda)\big) = (4\pi/b_0)e^{-\xi}$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}f_g(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{g\leftarrow q}(z)\sum_f\left(f_f\left(\frac{x}{z},\xi\right)+f_{\bar f}\left(\frac{x}{z},\xi\right)\right)\right.$$
$$\left.+\,P_{g\leftarrow g}(z)f_g\left(\frac{x}{z},\xi\right)\right], \tag{17.51}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}f_f(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z)f_f\left(\frac{x}{z},\xi\right)+P_{q\leftarrow g}(z)f_g\left(\frac{x}{z},\xi\right)\right], \tag{17.52}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}f_{\bar f}(x,\xi) = \frac{2}{b_0}\int_x^1 \frac{\mathrm{d}z}{z}\left[P_{q\leftarrow q}(z)f_{\bar f}\left(\frac{x}{z},\xi\right)+P_{q\leftarrow g}(z)f_g\left(\frac{x}{z},\xi\right)\right]. \tag{17.53}$$

(b) Now we apply the approximation that 1) gluon PDF dominates the integrand in the A-P equations and 2) the function $\tilde g(x,Q) = xf_g(x,Q)$ is slowly varying with x when x is small. Then, define $w = \log(1/x)$, which gives $\mathrm{d}/\mathrm{d}w = -x\mathrm{d}/\mathrm{d}x$, we can calculate

$$\begin{aligned}
\frac{\partial^2}{\partial w\partial\xi}\tilde g(x,\xi) &= -\,x\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\partial}{\partial\xi}f_g(x,Q)\right)\\
&\simeq -\,x\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{b_0}\int_x^1\frac{\mathrm{d}z}{z}\,P_{g\leftarrow g}(z)f_g\left(\frac{x}{z},Q\right)\right)\\
&= \frac{2}{b_0}\cdot xP_{g\leftarrow g}(x)f_g(x,Q) - \frac{2x}{b_0}\int_x^1 \mathrm{d}zP_{g\leftarrow g}(z)\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x}{z}f_g\left(\frac{x}{z},Q\right)\right]\\
&\simeq \frac{2}{b_0}\cdot xP_{g\leftarrow g}(x)f_g(x,Q).
\end{aligned} \tag{17.54}$$

From the result of Problem 17.4 we know that $xP_{g\leftarrow g}(x) = 6$ as $x\to 0$. Therefore the A-P equation for f_g becomes

$$\frac{\partial^2}{\partial w\partial\xi}\tilde g(x,\xi) = \frac{12}{b_0}\tilde g(x,\xi). \tag{17.55}$$

Then we verify that

$$\tilde{g} = K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) \tag{17.56}$$

is an approximation solution to the differential equation above when $w\xi \gg 1$, where $K(Q^2)$ is an initial condition. We apply $\partial^2/\partial w \partial \xi$ on this expression, to get

$$\begin{aligned} \frac{\partial^2}{\partial w \partial \xi} \tilde{g}(w, \xi) = & \frac{1}{4} \sqrt{\frac{48}{b_0 w(\xi - \xi_0)}} \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) \\ & \times \left[2(\xi - \xi_0) \frac{\partial K(Q^2)}{\partial \xi} + \left(1 + \sqrt{\frac{48}{b_0} w(\xi - \xi_0)}\right) K(Q^2)\right]. \end{aligned} \tag{17.57}$$

In the limit $w\xi \gg 1$, the square root term in the last line dominates, thus

$$\frac{\partial^2}{\partial w \partial \xi} \tilde{g}(w, \xi) \simeq \frac{b_0}{12} K(Q^2) \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) = \frac{12}{b_0} \tilde{g}(w, \xi). \tag{17.58}$$

(c) Then we consider the A-P equation for quarks. If we adopt the approximation in (b) again, namely, the gluon PDF dominates and the function $\tilde{q}(x, \xi) = x f_f(x, Q)$ is slowly varying, then we have

$$\begin{aligned} \frac{\partial}{\partial \xi} \tilde{q}(x, \xi) = x \frac{\partial}{\partial \xi} f_f(x, \xi) &= \frac{2x}{b_0} \int_x^1 \frac{\mathrm{d}z}{z} P_{q \leftarrow g}(z) f_g\Big(\frac{x}{z}, \xi\Big) \\ &= \frac{2}{b_0} \int_x^1 \mathrm{d}z P_{q \leftarrow g}(z) \tilde{g}\Big(\frac{x}{z}, \xi\Big) = \frac{1}{b_0} \int_x^1 \mathrm{d}z \big(z^2 + (1-z)^2\big) \tilde{g}\Big(\frac{x}{z}, \xi\Big) \\ &\simeq \frac{2}{b_0} \left[\frac{1}{6}(2z^3 - 2z^2 + 3z) \tilde{g}\Big(\frac{x}{z}, \xi\Big)\right]_x^1 \simeq \frac{2}{3b_0} \tilde{g}(x, \xi), \end{aligned} \tag{17.59}$$

where we have used $x \ll 1$ and $\partial \tilde{q}(x, \xi)/\partial x \simeq 0$. Then, we verify that

$$\tilde{q} = \sqrt{\frac{\xi - \xi_0}{27 b_o w}} K(Q^2) \cdot \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) \tag{17.60}$$

is again an approximate solution to the equation derived above, in the limit $w\xi \gg 1$. In fact,

$$\begin{aligned} \frac{\partial}{\partial \xi} \tilde{q}(x, \xi) = & \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) \left[\frac{2}{3b_0} K(Q^2) \right. \\ & \left. + \frac{1}{18} \sqrt{\frac{3}{b_0 w(\xi - \xi_0)}} \Big(K(Q^2) + 2(\xi - \xi_0) \frac{\partial K(Q^2)}{\partial \xi}\Big)\right] \\ \simeq & \frac{2}{3b_0} K(Q^2) \exp\left(\left[\frac{48}{b_0} w(\xi - \xi_0)\right]^{1/2}\right) = \frac{2}{3b_0} \tilde{g}(x, \xi). \end{aligned} \tag{17.61}$$

(d) We use the fitted formula of $K(Q^2)$ to plot the PDFs of gluon and quarks in Figure.

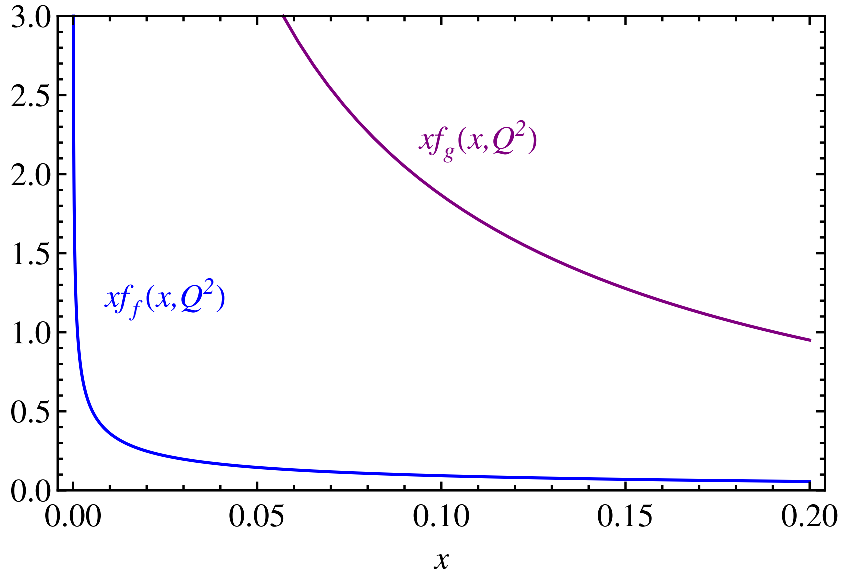


Figure 17.4: Approximate parton distribution functions at small x with $Q = 500\text{GeV}$.

Chapter 18

Operator Products and Effective Vertices

18.1 Matrix element for proton decay

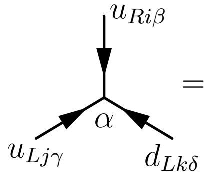
(a) We estimate the order of magnitude of the proton lifetime, through the decay $p \to e^+\pi^0$, based on the following operator,

$$\mathcal{O}_X = \frac{2}{m_X^2}\epsilon_{ijk}\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}e_{R\alpha}u_{Ri\beta}u_{Lj\gamma}d_{Lk\delta}, \tag{18.1}$$

where m_X is the scale of this higher dimensional operator, whose typical value is around 10^{16}GeV, and i, j, k are color indices for quarks, $\alpha, \beta, \cdots$ are spinor indices. Then, the amplitude $\mathcal{M}$ of this decay process should be proportional to m_X^{-2}. Note that the amplitude $\mathcal{M}$ has mass dimension 1, thus we should have $\mathcal{M} \sim m_p^3 m_X^{-2}$ with m_p the proton mass. Now take $m_X \sim 10^{16}$GeV and $m_p \sim 1$GeV, we have the decay width

$$\Gamma \sim \frac{1}{8\pi}\frac{1}{2m_p}|\mathcal{M}|^2 \sim \frac{1}{16\pi}\frac{m_p^5}{m_X^4} \sim 10^{-65}\text{GeV} \sim 10^{33}\text{yr}^{-1}. \tag{18.2}$$

(b) Now we consider the first order QCD correction to the estimation above. The correction comes from virtual gluon exchange among three quarks in the operator. To evaluate these 1-loop diagrams, we firstly fixed the renormalization condition of $\mathcal{O}_X$ to be



$$= \mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta}. \tag{18.3}$$

The 1-loop diagrams are shown in Figure 18.1. The Feynman rules can be written in two-component spinor notations. The left-handed spinor's propagator reads $\mathrm{i}(p\cdot\sigma)/p^2$, the right-handed spionr's propagator is $\mathrm{i}(p\cdot\bar{\sigma})/p^2$, the QCD interaction between quark and gluon is $\mathrm{i}[\psi_{Li}^\dagger\bar{\sigma}^\mu(t^a)_{ij}\psi_{Lj} + \psi_{Ri}^\dagger\sigma^\mu(t^a)_{ij}\psi_{Rj}]$, and the vertex corresponding $\mathcal{O}_X$ reads $\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta}$. Then

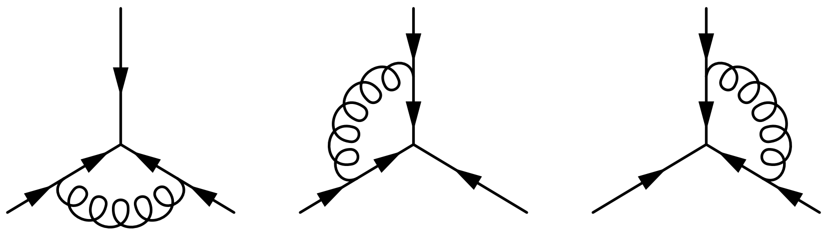


Figure 18.1: 1-loop QCD correction to the effective operator of proton decay.

the first diagram reads

$$\begin{aligned}
(\text{a}) =& \, \mathrm{i}(\mathrm{i}g)^2\epsilon_{imn}(t^a)_{mj}(t^a)_{nk}\delta^{\alpha\beta}\epsilon^{\gamma'\delta'}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{-\mathrm{i}}{q^2}\Big(\frac{-\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}_\mu\Big)_{\gamma'\gamma}\Big(\frac{\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}^\mu\Big)_{\delta'\delta}\\
=& - g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma'\delta'}(\sigma_\rho\bar{\sigma}_\mu)_{\gamma'}{}^{\gamma}(\sigma_\sigma\bar{\sigma}^\mu)_{\delta'}{}^{\delta}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{q^\rho q^\sigma}{q^6}\\
=& - g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\delta^{\alpha\beta}\cdot 16\epsilon^{\gamma\delta}\cdot\frac{\mathrm{i}}{4(4\pi)^2}\frac{2}{\epsilon}\\
=& \frac{8g^2}{3(4\pi)^2}\frac{2}{\epsilon}\cdot\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta},
\end{aligned} \tag{18.4}$$

where the Pauli matrices is simplified as follows,

$$\begin{aligned}
\epsilon^{\gamma'\delta'}(\sigma_\rho\bar{\sigma}_\mu)_{\gamma'}{}^{\gamma}(\sigma_\sigma\bar{\sigma}^\mu)_{\delta'}{}^{\delta} &= (\bar{\sigma}_\mu^T\sigma_\rho^T\epsilon\sigma^\rho\bar{\sigma}^\mu)^{\gamma\delta} = -(\bar{\sigma}_\mu^T\epsilon\sigma_\rho\sigma^\rho\bar{\sigma}^\mu)^{\gamma\delta}\\
&= -4(\bar{\sigma}_\mu^T\epsilon\bar{\sigma}^\mu)^{\gamma\delta} = -4(\epsilon\bar{\sigma}_\mu\bar{\sigma}^\mu)^{\gamma\delta} = 16\epsilon^{\gamma\delta},
\end{aligned} \tag{18.5}$$

in which we used the fact that $\epsilon^{\alpha\beta} = \mathrm{i}\sigma^2$ and $\sigma_\mu^T\sigma^2 = -\sigma^2\sigma_\mu$. In the computation of this diagram, we also used $\epsilon_{imn}(t^a)_{mj}(t^a)_{nk} = -(2/3)\epsilon_{ijk}$. The coefficient of this equality can be easily justified by contracting both sides with ϵ_{ijk}. Similarly, we compute the second diagram, as follows,

$$\begin{aligned}
(\text{b}) =& \, \mathrm{i}(\mathrm{i}g)^2\epsilon_{imn}(t^a)_{mj}(t^a)_{nk}\delta^{\alpha\beta'}\epsilon^{\gamma'\delta}\int\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{-\mathrm{i}}{q^2}\Big(\frac{-\mathrm{i}q\cdot\bar{\sigma}}{q^2}\sigma_\mu\Big)_{\beta'\beta}\Big(\frac{\mathrm{i}q\cdot\sigma}{q^2}\bar{\sigma}^\mu\Big)_{\gamma'\gamma}\\
=& - g^2\cdot\Big(-\frac{2}{3}\Big)\epsilon_{ijk}\cdot 4\delta^{\alpha\beta}\epsilon^{\gamma\delta}\cdot\frac{\mathrm{i}}{4(4\pi)^2}\frac{2}{\epsilon}\\
=& \frac{2g^2}{3(4\pi)^2}\frac{2}{\epsilon}\cdot\mathrm{i}\epsilon_{ijk}\delta^{\alpha\beta}\epsilon^{\gamma\delta},
\end{aligned} \tag{18.6}$$

where we used the identity $(\bar{\sigma}_\mu)_{\alpha\beta}(\sigma^\mu)_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma}$. The third diagram gives the same result as the second one. Therefore, we get the counterterm for the operator $\mathcal{O}_X$ in $\overline{MS}$ scheme to be

$$\delta_{\mathcal{O}_X} = -\frac{4g^2}{(4\pi)^2}\Big(\frac{2}{\epsilon} - \log M^2\Big), \tag{18.7}$$

where M^2 is the renormalization scale. We further recall that the field strength renormalization counterterm for quarks in QCD is given by

$$\delta_2 = -\frac{4g^2}{3(4\pi)^2}\Big(\frac{2}{\epsilon} - \log M^2\Big), \tag{18.8}$$

then the anomalous dimension of operator $\mathcal{O}_X$ is given by

$$\gamma = M\frac{\partial}{\partial M}\big(-\delta_{\mathcal{O}_X}+\frac{3}{2}\delta_2\big) = -\frac{4g^2}{(4\pi)^2}. \tag{18.9}$$

Therefore this QCD correction will enhance the operator strength by a factor of

$$\bigg(\frac{\log(m_X^2/\Lambda^2)}{\log(m_p^2/\Lambda^2)}\bigg)^{a_0/2b_0}, \tag{18.10}$$

where $\Lambda \simeq 200$GeV, $a_0 = 4$ is the coefficient from anomalous dimension, and $b_0 = 11 - (2/3)n_f = 7$ is the 1-loop coefficient of QCD β function. Taking $m_X = 10^{16}$GeV and $m_p = 1$GeV, this factor is about 2.5. Then the decay rate of proton is enhanced by a factor of $2.5^2 \simeq 6.3$.

18.2 Parity-violating deep inelastic form factor

(a) We firstly compute the amplitude of the neutrino deep inelastic scattering through charged current interaction, which reads

$$\begin{aligned} \mathrm{i}\mathcal{M}(\nu p\to\mu^- X) = {}& \frac{\mathrm{i}g^2}{2m_W^2}\bar{u}(k')\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)u(k) \\ &\times\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\langle X|\big(J_+^\mu(x)+J_-^\mu(x)\big)|P\rangle. \end{aligned} \tag{18.11}$$

Then the squared amplitude with initial proton's spins averaged and final state X summed is

$$\begin{aligned} \frac{1}{2}\sum|\mathcal{M}|^2 = {}& \frac{1}{2}\frac{g^4}{4m_W^4}\sum_{\text{spin}}\bigg(\bar{u}(k')\gamma_\nu\Big(\frac{1-\gamma_5}{2}\Big)u(k)\bar{u}(k)\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\mu u(k')\bigg) \\ &\times\sum_X\int \mathrm{d}\Pi_X\langle P|\big(J_+^\mu(x)+J_-^\mu(x)\big)|X\rangle\langle X|\big(J_+^\nu(0)+J_-^\nu(0)\big)|P\rangle. \end{aligned} \tag{18.12}$$

The trace factor can be straightforwardly worked out to be

$$\begin{aligned} L_{\mu\nu} \equiv{}& \sum_{\text{spin}}\bigg(\bar{u}(k')\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)u(k)\bar{u}(k)\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\nu u(k')\bigg) \\ ={}& \operatorname{tr}\Big[\gamma_\mu\Big(\frac{1-\gamma_5}{2}\Big)\not{k}\Big(\frac{1+\gamma_5}{2}\Big)\gamma_\nu\not{k}'\Big] \\ ={}& 2\big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma\big). \end{aligned} \tag{18.13}$$

Then, use the optical theorem, we have

$$\begin{aligned} & L_{\mu\nu}\sum_X\int \mathrm{d}\Pi_X\langle P|\big(J_+^\mu(x)+J_-^\mu(x)\big)|X\rangle\langle X|\big(J_+^\nu(0)+J_-^\nu(0)\big)|P\rangle \\ & = 2\,\mathrm{Im}\,\big(L_{\mu\nu}W^{\mu\nu(\nu)}\big), \end{aligned} \tag{18.14}$$

with

$$W^{\mu\nu(\nu)} = 2\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\langle P|T\big\{J^\mu_-(x)J^\nu_+(0)\big\}|P\rangle, \tag{18.15}$$

Therefore, the cross section is

$$\begin{aligned}\sigma(\nu p\to\mu^- X) &= \frac{1}{2s}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{1}{2k'}\cdot\frac{1}{2}\sum|\mathcal{M}|^2\\ &= \frac{1}{2s}\int \mathrm{d}x\mathrm{d}y\frac{ys}{(4\pi)^2}\cdot\frac{g^4}{4m_W^4}\,\mathrm{Im}\,\big(L_{\mu\nu}W^{\mu\nu(\nu)}\big),\end{aligned} \tag{18.16}$$

and the differential cross section is thus given by

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\mu^- X) = \frac{yG_F^2}{2\pi^2}\,\mathrm{Im}\,\big[\big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma\big)W^{\mu\nu(\nu)}\big]. \tag{18.17}$$

(b) The lepton momentum tensor obtained in (a) is

$$L_{\mu\nu} = 2(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}k\cdot k' + \mathrm{i}\epsilon_{\mu\nu}{}^{\rho\sigma}k_\rho k'_\sigma). \tag{18.18}$$

Then it is straightforward to see that $q^\mu L_{\mu\nu} = (k-k')^\mu L_{\mu\nu} = 0$ and $q^\nu L_{\mu\nu} = 0$. As a consequence, any term in $W^{\mu\nu(\nu)}$ proportional to q^μ or q^ν is irrelevant. Therefore we can rewrite the tensor $W^{\mu\nu(\nu)}$ in terms of three form factors $W_i^{(\nu)}$ $(i=1,2,3)$. That is,

$$W^{\mu\nu(\nu)} = -g^{\mu\nu}W_1^{(\nu)} + P^\mu P^\nu W_2^{(\nu)} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma W_3^{(\nu)} + \cdots. \tag{18.19}$$

Then the deep inelastic scattering cross section becomes

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\mu^- X) = \frac{yG_F^2}{2\pi^2}\Big[&2(k\cdot k')\,\mathrm{Im}\,W_1^{(\nu)} + 2(P\cdot k)(P\cdot k')\,\mathrm{Im}\,W_2^{(\nu)}\\ &-4\Big((P\cdot k)(q\cdot k') - (q\cdot k)(P\cdot k')\Big)\,\mathrm{Im}\,W_3^{(\nu)}\Big].\end{aligned} \tag{18.20}$$

(c) Now we evaluate $\mathrm{Im}\,W_{1,2,3}^{(\nu)}$ in the parton model. Firstly, $W^{\mu\nu(\nu)}$ can be written as

$$W^{\mu\nu(\nu)} = 2\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\int_0^1 \mathrm{d}\xi\sum_f f_f(\xi)\frac{1}{\xi}\big\langle q_f(\xi P)\big|T\{J^\mu_-(x)J^\nu_+(0)\}\big|q_f(\xi P)\big\rangle, \tag{18.21}$$

and be evaluated in terms of Feynman diagrams displayed in Fig. 18.10 of Peskin & Schroeder. For the first diagram, we have

$$\begin{aligned}2\mathrm{i}\int_0^1\mathrm{d}\xi\,\bigg[&f_d(\xi)\frac{1}{\xi}\bar u(p)\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not p+\not q+\mathrm{i}\epsilon}\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)u(p)\\ &+f_{\bar u}(\xi)\frac{1}{\xi}\bar u(p)\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not p+\not q+\mathrm{i}\epsilon}\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)u(p)\bigg],\end{aligned} \tag{18.22}$$

where $p=\xi P$. Then, averaging/summing over initial/final spin states gives

$$2\int_0^1\mathrm{d}\xi\,\bigg[f_d(\xi)\frac{1}{\xi}\cdot\frac{1}{2}\,\mathrm{tr}\,\Big(\not p\gamma^\mu\frac{1-\gamma^5}{2}(\not p+\not q)\gamma^\nu\frac{1-\gamma^5}{2}\Big)$$

$$+ f_{\bar{u}}(\xi)\frac{1}{\xi}\cdot\frac{1}{2}\,\mathrm{tr}\left(\not{p}\gamma^\mu\frac{1-\gamma^5}{2}(\not{p}+\not{q})\gamma^\nu\frac{1-\gamma^5}{2}\right)\bigg]\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}$$

$$\Rightarrow \int_0^1\frac{\mathrm{d}\xi}{\xi}\bigg[\big(f_d(\xi)+f_{\bar{u}}(\xi)\big)\big(4\xi^2P^\mu P^\nu-2\xi P\cdot q g^{\mu\nu}\big) + \big(f_d(\xi)-f_{\bar{u}}(\xi)\big)2\mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}P^\rho q^\sigma\bigg]\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}, \tag{18.23}$$

where we have dropped terms containing q^μ or q^ν in the last line. Then it is easy to read from this expression that

$$\mathrm{Im}\,W_1^{(\nu)} = 2P\cdot q\int_0^1\mathrm{d}\xi\,\big[f_d(\xi)+f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\Big(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\Big), \tag{18.24}$$

$$\mathrm{Im}\,W_2^{(\nu)} = \int_0^1\mathrm{d}\xi\,4\xi\big[f_d(\xi)+f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\Big(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\Big), \tag{18.25}$$

$$\mathrm{Im}\,W_3^{(\nu)} = 2\int_0^1\mathrm{d}\xi\,\big[f_d(\xi)-f_{\bar{u}}(\xi)\big]\,\mathrm{Im}\Big(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\Big), \tag{18.26}$$

where $2P\cdot q = ys$, and

$$\mathrm{Im}\Big(\frac{-1}{2p\cdot q+q^2+\mathrm{i}\epsilon}\Big) = \frac{\pi}{ys}\delta(\xi-x). \tag{18.27}$$

Note that the second diagram in Fig. 18.10 of Peskin & Schroeder does not contributes, as explained in the book. Therefore we conclude that

$$\mathrm{Im}\,W_1^{(\nu)} = \pi\big[f_d(x)+f_{\bar{u}}(x)\big], \tag{18.28}$$

$$\mathrm{Im}\,W_2^{(\nu)} = \frac{4\pi x}{ys}\big[f_d(x)+f_{\bar{u}}(x)\big], \tag{18.29}$$

$$\mathrm{Im}\,W_3^{(\nu)} = \frac{2\pi}{ys}\big[f_d(x)-f_{\bar{u}}(x)\big]. \tag{18.30}$$

(d) The analysis above can be easily repeated for the left-handed current J_{fL}^μ of single flavor f, defined by $J_{fL}^\mu = \bar{f}\gamma^\mu P_L f$ where $P_L \equiv (1-\gamma^5)/2$. Then, define

$$W_{fL}^{\mu\nu} = 2\mathrm{i}\int\mathrm{d}^4x\,e^{\mathrm{i}q\cdot x}\big\langle P\big|T\{J_{fL}^\mu(x)J_{fL}^\nu(0)\}\big|P\big\rangle, \tag{18.31}$$

and its decomposition,

$$W_{fL}^{\mu\nu} = -g^{\mu\nu}W_{1fL} + P^\mu P^\nu W_{2fL} + \mathrm{i}\epsilon^{\mu\nu\rho\sigma}P_\rho q_\sigma W_{3fL} + \cdots\,. \tag{18.32}$$

We see that it amounts to the replacement in the final result that $d\to f$ and $\bar{u}\to\bar{f}$. Therefore,

$$\mathrm{Im}\,W_{1fL}^{(\nu)} = \pi\big[f_f(x)+f_{\bar{f}}(x)\big], \tag{18.33}$$

$$\mathrm{Im}\,W_{2fL}^{(\nu)} = \frac{4\pi x}{ys}\big[f_f(x)+f_{\bar{f}}(x)\big], \tag{18.34}$$

$$\mathrm{Im}\,W_{3fL}^{(\nu)} = \frac{2\pi}{ys}\big[f_f(x)-f_{\bar{f}}(x)\big]. \tag{18.35}$$

(e) Now we perform OPE on $W_{fL}^{\mu\nu}$. Firstly,

$$\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x} J_{fL}^{\mu}(x) J_{fL}^{\nu}(0) \simeq \int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x}\Big(\bar{q}\gamma^\mu P_L \overline{q(x)\bar{q}}\gamma^\nu P_L q(0) + \bar{q}\gamma^\mu P_L q(x)\bar{q}\gamma^\nu P_L q(0)\Big)$$

$$= \bar{q}\gamma^\mu P_L \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^\nu P_L q + \Big(\mu\leftrightarrow\nu, q\to -q\Big). \tag{18.36}$$

Then, the first term in the last line can be written as

$$\bar{q}\gamma^\mu P_L \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^\nu P_L q = \frac{1}{2}\bigg(\bar{q}\gamma^\mu \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^\nu q - \bar{q}\gamma^\mu \frac{\mathrm{i}(\mathrm{i}\partial\!\!\!/ + q\!\!\!/)}{(\mathrm{i}\partial + q)^2}\gamma^\nu\gamma^5 q\bigg)$$

$$= -\frac{\mathrm{i}}{2}\bar{q}\big[2\gamma^{(\mu}(\mathrm{i}\partial^{\nu)}) - g^{\mu\nu}q\!\!\!/ - \mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}(\mathrm{i}\partial + q)^\rho\gamma^\sigma\big]$$

$$\times \frac{1}{Q^2}\sum_{n=0}^{\infty}\Big(\frac{2\mathrm{i}q\cdot\partial}{Q^2}\Big)^n q, \tag{18.37}$$

where we have symmetrize the $\mu\nu$ indices for the first two terms in the square bracket and antisymmetrize the indices for the third term, by using the equalities $\frac{1}{2}(\gamma^\mu\gamma^\lambda\gamma^\nu + \gamma^\nu\gamma^\lambda\gamma^\mu) = g^{\mu\lambda}\gamma^\nu + g^{\nu\lambda}\gamma^\mu - g^{\mu\nu}\gamma^\lambda$ and $\frac{1}{2}(\gamma^\mu\gamma^\lambda\gamma^\nu\gamma^5 - \gamma^\nu\gamma^\lambda\gamma^\mu\gamma^5) = -\mathrm{i}\epsilon^{\mu\nu\lambda}{}_\rho\gamma^\rho$, and terms proportional to q^μ or q^ν have also been dropped. The (anti)symmetrization can be understood by looking at (18.32), where the terms with no γ^5 are symmetric on $\mu\nu$ while the term involving γ^5 is antisymmetric on $\mu\nu$. Therefore, when including the second term in (18.36), we should keep terms of even powers in q for symmetric $\mu\nu$ indices and of odd powers in q for antisymmetric $\mu\nu$.

Now, with these understood, and using the definition of twist-2, spin-n operator,

$$\mathcal{O}_f^{(n)\mu_1\cdots\mu_n} = \bar{q}_f\gamma^{(\mu_1}(\mathrm{iD}^{\mu_2})\cdots(\mathrm{iD}^{\mu_n)})q_f - \text{traces}, \tag{18.38}$$

we have,

$$\mathrm{i}\int \mathrm{d}^4x\, e^{\mathrm{i}q\cdot x} J_{fL}^{\mu}(x) J_{fL}^{\nu}(0) = \sum_{n>0,\text{ even}} 2\frac{(2q_{\mu_1})\cdots(2q_{\mu_{n-2}})}{(Q^2)^{n-1}}\mathcal{O}_f^{(n)\mu\nu\mu_1\cdots\mu_{n-2}}$$

$$-\frac{1}{2}g^{\mu\nu}\sum_{n>0,\text{ even}}\frac{(2q_{\mu_1})\cdots(2q_{\mu_n})}{(Q^2)^n}\mathcal{O}_f^{(n)\mu_1\cdots\mu_n}$$

$$-\mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}q^\rho\sum_{n>0\text{ odd}}\frac{(2q_{\mu_1})\cdots(2q_{\mu_{n-1}})}{(Q^2)^n}\mathcal{O}_f^{(n)\sigma\mu_1\cdots\mu_{n-1}}. \tag{18.39}$$

Then, using $\langle P|\mathcal{O}_f^{(n)\mu_1\cdots\mu_n}|P\rangle = 2A_f^n P^{\mu_1}\cdots P^{\mu_n}$, we can get $W_{fL}^{\mu\nu}$ to be

$$W_{fL}^{\mu\nu} = 8P^\mu P^\nu\sum_{n>0,\text{ even}}\frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-1}}A_f^n - 2g^{\mu\nu}\sum_{n>0,\text{ even}}\frac{(2q\cdot P)^n}{(Q^2)^n}A_f^n$$

$$+ 4\mathrm{i}\epsilon^{\mu\nu}{}_{\rho\sigma}P^\rho q^\sigma\sum_{n>0,\text{ odd}}\frac{(2q\cdot P)^{n-1}}{(Q^2)^n}A_f^n. \tag{18.40}$$

So we can read out

$$W_{1fL} = 2 \sum_{n>0,\text{ even}} \frac{(2q\cdot P)^n}{(Q^2)^n} A_f^n, \tag{18.41}$$

$$W_{2fL} = 8 \sum_{n>0,\text{ even}} \frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-1}} A_f^n, \tag{18.42}$$

$$W_{3fL} = 4 \sum_{n>0,\text{ odd}} \frac{(2q\cdot P)^{n-1}}{(Q^2)^n} A_f^n. \tag{18.43}$$

(f) Now we use W_{3fL} obtained above to derive a sum rule for parton distribution f_f^-, defined by

$$f_f^-(x, Q^2) = \frac{ys}{2\pi}\,\mathrm{Im}\,W_{3fL}(x, Q^2), \tag{18.44}$$

where $x = Q^2/\nu$ and $\nu = 2P\cdot q = ys$. The the analytic behavior of W_{3fL} on the v-complex plane is shown in Fig. 18.11 of Peskin & Schroeder. Thus we can define the contour integral

$$I_{3n} = \int \frac{\mathrm{d}\nu}{2\pi\mathrm{i}} \frac{1}{\nu^n} W_{3fL}(\nu, Q^2), \tag{18.45}$$

where the contour is a small circle around the origin $\nu = 0$. This integral picks up the coefficient of $\nu^{(n-1)}$ term, namely, $I_{3n} = 4A_f^n/(Q^2)^n$. On the other hand, the contour can be deformed as shown in Fig. 18.12 of Peskin & Schroeder. Then the integral can be evaluated as

$$I_{3n} = 2\int_{Q_f^2}^{\infty} \frac{\mathrm{d}\nu}{2\pi\mathrm{i}} \frac{1}{\nu^n}(2\mathrm{i})\,\mathrm{Im}\,W_{3fL}(\nu, Q^2) = \frac{4}{(Q^2)^n}\int_0^1 \mathrm{d}x\, x^{n-1} f_f^-(x, Q^2). \tag{18.46}$$

Therefore we get the sum rule,

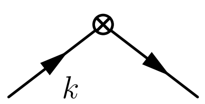
$$\int_0^1 \mathrm{d}x\, x^{n-1} f_f^-(x, Q^2) = A_f^n. \tag{18.47}$$

18.3 Anomalous dimensions of gluon twist-2 operators

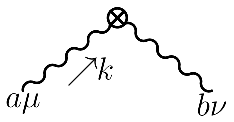
In this problem we finish evaluating anomalous dimension matrix γ^n in (18.180) of Peskin & Schroeder, given by

$$\gamma^n = -\frac{g^2}{(4\pi)^2}\begin{pmatrix} a_{ff}^n & a_{fg}^n \\ a_{gf}^n & a_{gg}^n \end{pmatrix} \tag{18.48}$$

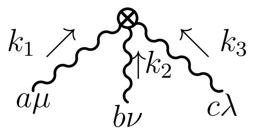
where a_{ff}^n has already been evaluated explicitly in the book. Here we evaluate the remaining three elements. The needed Feynman rules involving operators $\mathcal{O}_f^{(n)}$ and $\mathcal{O}_g^{(n)}$ are listed as follows:



$$= \not{\Lambda}(\Lambda\cdot k)^{n-1},$$



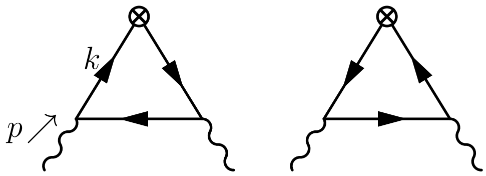
$$= -2\Big[g^{\mu\nu}(\Lambda\cdot k)^n + k^2\Lambda^\mu\Lambda^\nu(\Lambda\cdot k)^{n-2} - 2k^{(\mu}\Lambda^{\nu)}(\Lambda\cdot k)^{n-1}\Big]\delta^{ab},$$



$$\begin{aligned}=&-2\mathrm{i}gf^{abc}g^{\mu\nu}\Lambda^\lambda\sum_{j=1}^{n}(\Lambda\cdot k_1)^{j-1}(-\Lambda\cdot k_2)^{n-j}\\ &+(\text{cyclic permutations on }\mu ak_1,\nu bk_2,\lambda ck_3)+\cdots.\end{aligned}$$

In the last expression, we list only terms containing a metric tensor $g^{\mu\nu}$, and the ignored terms (marked by $\cdots$) are irrelevant in the following calculations. To be clear, we have introduced a source $J^{(n)}$ to these operators, namely, we write $\Delta\mathcal{L} = J^{(n)}_{\mu_1\cdots\mu_n}\mathcal{O}^{(n)\mu_1\cdots\mu_n}_{f,g}$, with $J^{(n)}_{\mu_1\cdots\mu_n} = \Lambda_{\mu_1}\cdots\Lambda_{\mu_n}$, and $\Lambda^2 = 0$. As can be easily seen, this source automatically projects the operator $\mathcal{O}^{(n)}_{f,g}$ to its symmetric and traceless part.

(a) Firstly, we consider a^n_{fg}, which can be got by evaluating the following two diagrams.



With the Feynman rules listed above, the first diagram reads,

$$\begin{aligned}&(\mathrm{i}g)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}(-1)\,\mathrm{tr}\Big[t^b\gamma^\nu\frac{\mathrm{i}}{\not{k}}\not{\Lambda}\frac{\mathrm{i}}{\not{k}}t^a\gamma^\mu\frac{\mathrm{i}}{\not{k}-\not{p}}\Big](\Lambda\cdot k)^{n-1}\\ =&-\mathrm{i}g^2\,\mathrm{tr}\,[t^at^b]\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2(1-x)}{(k'^2-\Delta)^3}(\Lambda\cdot k)^{n-1}\,\mathrm{tr}\,[\gamma^\nu\not{k}\not{\Lambda}\not{k}\gamma^\mu(\not{k}-\not{p})].\end{aligned}\tag{18.49}$$

We need to extract terms of proportional to $g^{\mu\nu}(\Lambda\cdot p)^n$ and of logarithmical divergence. This needs some manipulations on the numerator of the integrand. We firstly evaluate the gamma trace, keep terms containing at least two powers of k, and shift the variable $k^\mu \to k'^\mu = k^\mu - xp^\mu$. Then we pick up terms containing two k', which contributes to logarithmical divergence. At last we symmetrize the indices according to $k'^\mu k'^\nu \to k'^2g^{\mu\nu}/4$. The detailed steps are given as follows.

$$\begin{aligned}&(\Lambda\cdot k)^{n-1}\,\mathrm{tr}\,[\gamma^\nu\not{k}\not{\Lambda}\not{k}\gamma^\mu(\not{k}-\not{p})]\\ \Rightarrow&\Big[16(\Lambda\cdot k)^nk^\mu k^\nu\Big]-\Big[4(\Lambda\cdot k)^n(k-2p)\cdot kg^{\mu\nu}\Big]-\Big[4(\Lambda\cdot k')^{n-1}(\Lambda\cdot p)k^2g^{\mu\nu}\Big]\\ \Rightarrow&\Big[16x^n(\Lambda\cdot p)^nk'^\mu k'^\nu\Big]-\Big[4nx^n(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}\\ &\quad+4n(x-2)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}+4x^n(\Lambda\cdot p)^nk'^2g^{\mu\nu}\Big]\\ &-\Big[8(n-1)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}g^{\mu\nu}+4x^{n-1}(\Lambda\cdot p)^nk'^2g^{\mu\nu}\Big]\\ \Rightarrow&\Big[4x^n\Big](\Lambda\cdot p)^nk'^2g^{\mu\nu}-\Big[nx^n+n(x-2)x^{n-1}+4x^n\Big](\Lambda\cdot p)^nk'^2g^{\mu\nu}\end{aligned}$$

$$
\begin{aligned}
&- \Big[2(n-1)x^{n-1} + 4x^{n-1}\Big](\Lambda\cdot p)^n k'^2 g^{\mu\nu} \\
=& -(2nx^n + 2x^{n-1})(\Lambda\cdot p)^n k'^2 g^{\mu\nu}. \qquad (18.50)
\end{aligned}
$$

Then it is straightforward to finish the loop integral,

$$
\begin{aligned}
&\mathrm{i}g^2\,\mathrm{tr}\,[t^a t^b](\Lambda\cdot p)^n g^{\mu\nu}\int_0^1 \mathrm{d}x\, 2(1-x)(2nx^n+2x^{n-1})\int \frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{k'^2}{(k'^2-\Delta)^3} \\
=& -\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(\Lambda\cdot p)^n\delta^{ab}g^{\mu\nu}. \qquad (18.51)
\end{aligned}
$$

The second diagram contributes an identical term for n even. The two diagrams sum to

$$
\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\cdot\Big(-2(\Lambda\cdot p)^n\delta^{ab}g^{\mu\nu}\Big). \qquad (18.52)
$$

Therefore the corresponding counterterm reads

$$
\delta_{fg} = -\frac{g^2}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n+1)(n+2)}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}}, \qquad (18.53)
$$

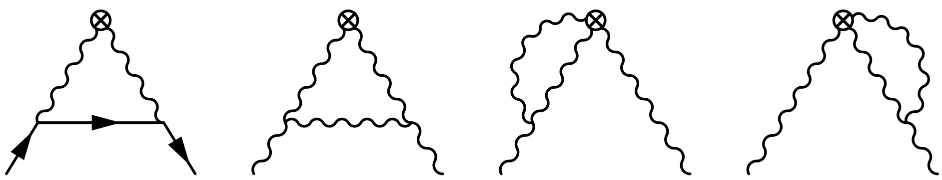
and the anomalous dimension element reads

$$
\gamma_{fg}^n = -M\frac{\partial}{\partial M}\delta_{fg} = -\frac{g^2}{(4\pi)^2}\frac{4(n^2+n+2)}{n(n+1)(n+2)}, \qquad (18.54)
$$

and thus,

$$
a_{fg}^n = \frac{4(n^2+n+2)}{n(n+1)(n+2)}. \qquad (18.55)
$$

(b) Then we consider a_{gf}^n and a_{gg}^n. This time we need to evaluate the following four diagrams.



The first diagram contributes to a_{gf}^n, which reads

$$
\begin{aligned}
&-2(\mathrm{i}g)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}t^a\gamma^\nu\frac{\mathrm{i}}{\not{p}-\not{k}}t^a\gamma^\mu\Big(\frac{\mathrm{i}}{k^2}\Big)^2 \\
&\times\Big[g_{\mu\nu}(\Lambda\cdot k)^n + k^2\Lambda_\mu\Lambda_\nu(\Lambda\cdot k)^{n-2} - 2k_{(\mu}\Lambda_{\nu)}(\Lambda\cdot k)^{n-1}\Big] \\
=& -2\mathrm{i}g^2C_2(N)\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2(1-x)}{(k'^2-\Delta)^3}\Big[\gamma^\mu(\not{p}-\not{k})\gamma_\mu(\Lambda\cdot k)^n \\
&+\not{\Lambda}(\not{p}-\not{k})\not{\Lambda}(\Lambda\cdot k)^{n-2}k^2 - \Big(\not{\Lambda}(\not{p}-\not{k})\not{k}+\not{k}(\not{p}-\not{k})\not{\Lambda}\Big)(\Lambda\cdot k)^{n-1}\Big] \qquad (18.56)
\end{aligned}
$$

To find the pieces proportional to $\not\Lambda(\Lambda\cdot p)^{n-1}$ and of logarithmical divergence, we manipulate on the expression in the square bracket, shifting the variable $k^\mu \to k'^\mu = k^\mu - xp^\mu$, extracting terms with two factors of k'^μ, symmetrizing the integrand with $k'^\mu k'^\nu \to k'^2 g^{\mu\nu}/4$, and throwing away terms proportional to $\Lambda^2(=0)$. This gives

$$\begin{aligned}
&\Big[\gamma^\mu(\not p-\not k)\gamma_\mu(\Lambda\cdot k)^n\Big] + \Big[\not\Lambda(\not p-\not k)\not\Lambda(\Lambda\cdot k)^{n-2}k^2\Big] \\
&\quad - \Big[\Big(\not\Lambda(\not p-\not k)\not k + \not k(\not p-\not k)\not\Lambda\Big)(\Lambda\cdot k)^{n-1}\Big] \\
&= \Big[-2(\not p-\not k)(\Lambda\cdot k)^n\Big] + \Big[2\not\Lambda\big(\Lambda\cdot(p-k)\big)(\Lambda\cdot k)^{n-2}k^2\Big] \\
&\quad - \Big[2\Big(\big(\Lambda\cdot(p-k)\big)\not k + \big((p-k)\cdot k\big)\not\Lambda - (\not p-\not k)(\Lambda\cdot k)\Big)(\Lambda\cdot k)^{n-1}\Big] \\
&\Rightarrow \Big[2nx^{n-1}\not k'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}\Big] + \Big[-2x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\quad + 2(n-2)(1-x)x^{n-2}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\quad + (1-x)x^{n-2}(\Lambda\cdot p)^{n-1}k'^2\Big] + \Big[2x^{n-1}\not k'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1} \\
&\quad - 2(n-1)(1-x)x^{n-2}(\Lambda\cdot k')(\Lambda\cdot p)^{n-1} - 2\not\Lambda\Big(-x^{n-1}k'^2(\Lambda\cdot p)^{n-1} \\
&\quad - (n-1)x^{n-1}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2} \\
&\quad + (n-1)(1-x)x^{n-2}(k'\cdot p)(\Lambda\cdot k')(\Lambda\cdot p)^{n-2}\Big) - 2nx^{n-1}\not k'(\Lambda\cdot k')(\Lambda\cdot p)^{n-1}\Big] \\
&\Rightarrow \Big[\frac{n}{2}x^{n-1}\Big]\not\Lambda k'^2(\Lambda\cdot p)^{n-1} + \Big[-x^{n-1} + n(1-x)x^{n-2}\Big]\not\Lambda k'^2(\Lambda\cdot p)^{n-1} \\
&\quad + \Big[-(n-1)(1-x)x^{n-2} + 2x^{n-1}\Big]\not\Lambda k'^2(\Lambda\cdot p)^{n-1} \\
&= \Big[\frac{n}{2}x^{n-1} + x^{n-2}\Big]\not\Lambda k'^2(\Lambda\cdot p)^{n-1}.
\end{aligned} \tag{18.57}$$

Then we have

$$\begin{aligned}
&-2\mathrm{i}g^2C_2(N)\not\Lambda(\Lambda\cdot p)^{n-1}\int_0^1 \mathrm{d}x\, 2(1-x)\Big(\frac{n}{2}x^{n-1}+x^{n-2}\Big)\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{k'^2}{(k'^2-\Delta)^3} \\
&= \frac{g^2C_2(N)}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n^2-1)}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}\not\Lambda(\Lambda\cdot p)^{n-1},
\end{aligned} \tag{18.58}$$

which gives the counterterm coefficient,

$$\delta^n_{gf} = -\frac{g^2C_2(N)}{(4\pi)^2}\frac{2(n^2+n+2)}{n(n^2-1)}\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}} \tag{18.59}$$

Then, in a similar way as in (a), we get

$$\gamma^n_{gf} = -\frac{g^2C_2(N)}{(4\pi)^2}\frac{4(n^2+n+2)}{n(n^2-1)}, \tag{18.60}$$

and for $N=3$, $C_2(N)=4/3$, we get

$$a^n_{gf} = \frac{16}{3}\frac{(n^2+n+2)}{n(n^2-1)}. \tag{18.61}$$

The second to fourth diagrams contribute to a_{gg}^n. Now we evaluate them in turn. The second one reads,

$$\begin{aligned}
&-2g^2 f^{ace} f^{bde}\delta^{cd}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Big(\frac{-\mathrm{i}}{k^2}\Big)^2\frac{-\mathrm{i}}{(p-k)^2}\\
&\times\Big[g^{\mu\rho}(p+k)^\lambda+g^{\rho\lambda}(p-2k)^\mu+g^{\lambda\mu}(k-2p)^\rho\Big]\\
&\times\Big[-g^{\nu\sigma}(p+k)_\lambda+g^\sigma_\lambda(2k-p)^\nu+g^\nu_\lambda(2p-k)^\sigma\Big]\\
&\times\Big[g_{\rho\sigma}(\Lambda\cdot k)^n+k^2\Lambda_\rho\Lambda_\sigma(\Lambda\cdot k)^{n-2}-2k_{(\rho}\Lambda_{\sigma)}(\Lambda\cdot k)^{n-1}\Big]\\
\Rightarrow&-2\mathrm{i}g^2C_2(G)\delta^{ab}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\,\frac{2(1-x)}{(k'^2-\Delta)^3}\Big[-8(\Lambda\cdot k)^nk^\mu k^\nu-\Big((\Lambda\cdot k)^nk^2\\
&+2(\Lambda\cdot k)^n(k\cdot p)-8(\Lambda\cdot k)^{n-1}(\Lambda\cdot p)(k\cdot p)+4(\Lambda\cdot k)^{n-2}(\Lambda\cdot p)^2k^2\Big)g^{\mu\nu}\Big]\\
\Rightarrow&-2\mathrm{i}g^2C_2(G)g^{\mu\nu}\delta^{ab}(\Lambda\cdot p)^n\int_0^1\mathrm{d}x\,2(1-x)\Big[-\Big(3+\frac{n}{2}\Big)x^n-\frac{1}{2}nx^{n-1}-2x^{n-2}\Big]\\
&\times\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{k'^2}{(k'^2-\Delta)^3}\\
=&-\frac{g^2C_2(G)}{(4\pi)^2}\Big(\frac{4}{n+2}-\frac{6}{n+1}+\frac{4}{n}-\frac{4}{n-1}\Big)\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(-2)g^{\mu\nu}\delta^{ab}(\Lambda\cdot p)^n. \qquad (18.62)
\end{aligned}$$

The third diagram reads (where an additional 1/2 is the symmetry factor),

$$\begin{aligned}
&-\frac{1}{2}\cdot 2\mathrm{i}g^2 f^{acd}f^{bcd}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(p-k)^2}\\
&\times\Big[g^{\mu\rho}(p+k)^\sigma+g^{\rho\sigma}(p-2k)^\mu+g^{\sigma\mu}(k-2p)^\rho\Big]\\
&\times\sum_{j=1}^n\Big[g^\nu_\sigma\Lambda_\rho(\Lambda\cdot(p-k))^{j-1}(\Lambda\cdot p)^{n-j}-g^\nu_\rho\Lambda_\sigma(\Lambda\cdot p)^{j-1}(\Lambda\cdot k)^{n-j}\Big]\\
\Rightarrow&-\mathrm{i}g^2C_2(G)g^{\mu\nu}\delta^{ab}(\Lambda\cdot p)^n\sum_{j=1}^n\int_0^1\mathrm{d}x\,\Big[(1+x)x^{n-j}-(x-2)(1-x)^{j-1}\Big]\\
&\times\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\frac{1}{(k'^2-\Delta)^2}\\
=&\frac{g^2C_2(G)}{(4\pi)^2}g^{\mu\nu}\delta^{ab}\frac{\Gamma(2-\frac{2}{d})}{\Delta^{2-d/2}}(\Lambda\cdot p)^n\\
&\times\sum_{j=1}^n\Big(\frac{1}{j}+\frac{1}{j+1}+\frac{1}{n-j+1}+\frac{1}{n-j+2}\Big)\\
\Rightarrow&-\frac{g^2C_2(G)}{(4\pi)^2}\Big[2\sum_{j=2}^n\frac{1}{j}+\frac{1}{n+1}+1\Big]\frac{\Gamma(2-\frac{2}{d})}{\Delta^{2-d/2}}(-2)g^{\mu\nu}(\Lambda\cdot p)^n. \qquad (18.63)
\end{aligned}$$

The contribution from fourth diagram is identical to the one from the third diagram. Summing the last three diagram together, we get

$$\frac{g^2C_2(G)}{(4\pi)^2}\Big(\frac{4}{(n+1)(n+2)}+\frac{4}{n(n+1)}-4\sum_{j=2}^n\frac{1}{j}-2\Big)\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}}(-2)g^{\mu\nu}\delta^{ab}(\Lambda\cdot p)^n. \quad (18.64)$$

Thus the corresponding counterterm is

$$\delta_g = -\frac{g^2 C_2(G)}{(4\pi)^2}\bigg(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - 2\bigg)\frac{\Gamma(2-\frac{d}{2})}{(M^2)^{2-d/2}}. \tag{18.65}$$

As a result,

$$\begin{aligned}\gamma_{gg}^n &= M\frac{\partial}{\partial M}(-\delta_g + \delta_3)\\ &= -\frac{2g^2}{(4\pi)^2}\bigg[\bigg(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - \frac{1}{3}\bigg)C_2(G) - \frac{4}{3}n_f C(N)\bigg],\end{aligned} \tag{18.66}$$

therefore, for $N=3$, $C_2(N) = 4/3$ and $C(N) = 1/2$, we have,

$$a_{gg}^n = 6\bigg(\frac{4}{(n+1)(n+2)} + \frac{4}{n(n+1)} - 4\sum_{j=2}^{n}\frac{1}{j} - \frac{1}{3} - \frac{2}{9}n_f\bigg). \tag{18.67}$$

18.4 Deep inelastic scattering from a photon

(a) The A-P equation for parton distributions in the photon can be easily written down by using the QED splitting functions listed in (17.121) of Peskin & Schroeder. Taking account of quarks' electric charge properly, we have,

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_q(x,Q) = \frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\bigg\{P_{e\leftarrow e}(z)f_q\Big(\frac{x}{z},Q\Big) + P_{e\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\bigg\}, \tag{18.68}$$

$$\frac{\mathrm{d}}{\mathrm{d}\log Q}f_{\bar q}(x,Q) = \frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\bigg\{P_{e\leftarrow e}(z)f_{\bar q}\Big(\frac{x}{z},Q\Big) + P_{e\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\bigg\}, \tag{18.69}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\log Q}f_\gamma(x,Q) = \sum_q\frac{3Q_q^2\alpha}{\pi}\int_x^1\frac{\mathrm{d}z}{z}\bigg\{&P_{\gamma\leftarrow e}(z)\Big[f_q\Big(\frac{x}{z},Q\Big) + f_{\bar q}\Big(\frac{x}{z},Q\Big)\Big]\\ &+ P_{\gamma\leftarrow\gamma}(z)f_\gamma\Big(\frac{x}{z},Q\Big)\bigg\},\end{aligned} \tag{18.70}$$

where the splitting functions are

$$P_{e\leftarrow e}(z) = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z), \tag{18.71}$$

$$P_{\gamma\leftarrow e}(z) = \frac{1+(1-z)^2}{z}, \tag{18.72}$$

$$P_{e\leftarrow\gamma}(z) = z^2 + (1-z)^2, \tag{18.73}$$

$$P_{\gamma\leftarrow\gamma}(z) = -\frac{2}{3}\delta(1-z). \tag{18.74}$$

We take $q = u,d,c,s$, and $Q_{u,c} = +2/3$, $Q_{d,s} = -1/3$. The factor 3 in the A-P equations above takes account of 3 colors. Since no more leptons appear in final states other than original e^+e^-, they are not included in the photon structure. With the initial condition

$f_\gamma(x,Q_0)=\delta(1-x)$ and $f_{q,\bar q}(x,Q_0)=0$ where $Q_0=0.5\text{GeV}$, these distribution functions can be solved from the equations above to the first order in α, to be

$$f_q(x,Q)=f_{\bar q}(x,Q)=\frac{3Q_q^2\alpha}{2\pi}\log\frac{Q^2}{Q_0^2}\big[x^2+(1-x)^2\big], \tag{18.75}$$

$$f_\gamma(x,Q)=\Big(1-\sum_q\frac{Q_q^2\alpha}{\pi}\log\frac{Q^2}{Q_0^2}\Big)\delta(1-x). \tag{18.76}$$

(b) The formulation of deep inelastic scattering from a photon is similar to the one for the proton, as described in Peskin & Schroeder. The process can be formulated as a two-photon scattering, with one photon being hard and the other one play the role of proton, which has the internal structure as shown in (a). Therefore, we can write down the corresponding current product as

$$W^{\mu\nu}=\mathrm{i}\int\mathrm{d}^4x\,e^{\mathrm{i}q\cdot x}\langle\gamma|\mathrm{T}\{J^\mu(x)J^\nu(0)\}|\gamma\rangle, \tag{18.77}$$

which can be again expanded in terms of scalar form factors,

$$W^{\mu\nu}=\Big(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\Big)W_1+\Big(P^\mu-q^\mu\frac{P\cdot q}{q^2}\Big)\Big(P^\nu-q^\nu\frac{P\cdot q}{q^2}\Big)W_2. \tag{18.78}$$

After operator product expansion, the form factor W_2 can be expressed as

$$W_2=3\sum_q Q_q^2\sum_n\frac{8}{Q^2}\frac{(2q\cdot P)^{n-2}}{(Q^2)^{n-2}}A_q^n(Q^2), \tag{18.79}$$

and $A_q^n(Q^2)$ is a scale-dependent quantity, whose scaling behavior is dictated by the anomalous dimension matrix γ. This matrix can be evaluated again from the diagrams in Fig. 18.13 in Peskin & Schroeder and in figures of last problem. The only difference is that we should replace the gluon field with photon field. Therefore it is straightforward to see that $a_{\gamma\gamma}^n=0$. For a_{qq}^n and $a_{\gamma q}^n$, we should take away the group factor $C_2(N)=4/3$, while for $a_{q\gamma}^n$, we should take away the factor $\mathrm{tr}\,(t^at^b)=\delta^{ab}/2$. In addition, we should also include the factor Q_q^2 corresponding to electric charge of each quark. Then we have,

$$a_{qq}^n=-2Q_f^2\Big[1+4\sum_{j=2}^n\frac{1}{j}-\frac{2}{n(n+1)}\Big], \tag{18.80}$$

$$a_{q\gamma}^n=\frac{8Q_f^2(n^2+n+2)}{n(n+1)(n+2)}, \tag{18.81}$$

$$a_{\gamma q}^n=\frac{4Q_f^2(n^2+n+2)}{n(n^2-1)}, \tag{18.82}$$

$$a_{\gamma\gamma}^n=0. \tag{18.83}$$

(c) The $n=2$ moment photon structure function can be worked out through the moment sum rules (18.154) in Peskin & Schroeder, where the matrix elements A_q^n in our case is a scale-dependent quantity. This dependence can be found by evaluating the anomalous dimension matrix of operator $\mathcal{O}_q^{(2)}$ as is done below (18.185) of Peskin & Schroeder, but with different entries, given by

$$\gamma = -\frac{\alpha}{4\pi}\begin{pmatrix} a_{uu}^2 & 0 & 3\times 2a_{u\gamma}^2 \\ 0 & a_{dd}^2 & 3\times 2a_{u\gamma}^2 \\ a_{\gamma u}^2 & a_{\gamma d}^2 & a_{\gamma\gamma}^2 \end{pmatrix} = -\frac{\alpha}{4\pi}\begin{pmatrix} -\frac{64}{27} & 0 & \frac{32}{27} \\ 0 & -\frac{16}{27} & \frac{8}{27} \\ \frac{64}{27} & \frac{16}{27} & 0 \end{pmatrix}. \tag{18.84}$$

(d) As can be inferred from (a), the photon structure function $f_\gamma(x, Q)$ is originally peaked at $x = 1$ for $Q = Q_0$, and the peak shifts toward smaller x and the peak goes lower and broader, as Q goes large from Q_0.

Chapter 19

Perturbation Theory Anomalies

19.1 Fermion number nonconservation in parallel E and B fields

(a) In this problem we investigate the effect of chiral anomaly on the (non)conservation of fermion number with definite chirality. Let us begin with the Adler-Bell-Jackiw anomaly equation,

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{19.1}$$

Integrating the left hand side over the whole spacetime, we get the difference between the numbers of right-handed fermions N_R and of left-handed fermions N_L, namely,

$$\int \mathrm{d}^4x\,\partial_\mu j^{\mu 5} = \int \mathrm{d}^4x\,\partial_\mu(j_R^\mu - j_L^\mu) = \int \mathrm{d}^3x\,(j_R^0 - j_L^0)\Big|_{t_1}^{t_2} = \Delta N_R - \Delta N_L, \tag{19.2}$$

where we assume that the integral region for time is $[t_1, t_2]$ and that $\partial_i j^i$ integrates to zero with suitable boundary conditions (i.e. vanishing at spatial infinity or periodic boundary condition). On the other hand,

$$\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = 4\epsilon^{0ijk}F_{0i}F_{jk} = -8F_{0i}\big(\tfrac{1}{2}\epsilon_{ijk}F_{jk}\big) = -8\mathbf{E}\cdot\mathbf{B}. \tag{19.3}$$

Therefore, the ABJ anomaly equation gives,

$$\Delta N_R - \Delta N_L = \frac{e^2}{2\pi^2}\int \mathrm{d}^4x\,\mathbf{E}\cdot\mathbf{B}. \tag{19.4}$$

(b) The Hamiltonian for massless charged fermions with background electromagnetic field is given by

$$H = \int \mathrm{d}^3x\,\big(\pi \mathrm{D}_0\psi - \mathcal{L}\big) = -\int \mathrm{d}^3x\,\mathrm{i}\bar{\psi}\gamma^i\mathrm{D}_i\psi, \tag{19.5}$$

where $\pi = \mathrm{i}\psi^\dagger$ is the canonical conjugate momentum of ψ, $\mathcal{L} = \mathrm{i}\bar{\psi}\not{D}\psi$ is the Lagrangian for the fermion, and $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}eA_\mu$ is the covariant derivative. Now we expand the Hamiltonian in the chiral basis,

$$
\begin{aligned}
H &= -\int \mathrm{d}^3x \begin{pmatrix} \psi_L^\dagger & \psi_R^\dagger \end{pmatrix} \begin{pmatrix} -\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D} & 0 \\ 0 & \mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D} \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= \int \mathrm{d}^3x \Big[\psi_L^\dagger(\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D})\psi_L - \psi_R^\dagger(\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D})\psi_R\Big].
\end{aligned}
\tag{19.6}
$$

(c) Now we focus on the eigenvalue problem of the right-handed fermion ψ_R, namely the equation $-\mathrm{i}\boldsymbol{\sigma}\cdot\mathbf{D}\psi_R = E\psi_R$. To be definite, we set the background electromagnetic potential to be $A^\mu = (0, 0, Bx^1, A)$ with B and A two constants. To seek the eigenfunction of the form $\psi_R = \big(\phi_1(x^1), \phi_2(x^1)\big)^T e^{\mathrm{i}(k_2x^2+k_3x^3)}$, we substitute it into the equation above and get

$$
\phi_1' = (k_2 - eBx^1)\phi_1 + \mathrm{i}(E + k_3 - eA)\phi_2, \tag{19.7a}
$$

$$
\phi_2' = \mathrm{i}(E - k_3 + eA)\phi_1 - (k_2 - eBx^1)\phi_2. \tag{19.7b}
$$

Eliminating ϕ_2 from these two equations, we get a single differential equation in the form of the harmonic oscillator,

$$
\phi_1'' - \Big[e^2B^2\Big(x^1 - \frac{k_2}{eB}\Big)^2 - E^2 + (k_3 - eA)^2 - eB\Big]\phi_1 = 0. \tag{19.8}
$$

(d) Now we specify the spatial boundary condition to be the box of length L in each side and periodic boundary condition. Then the condition $\psi_R(x^1, x^2, x^3) = \psi_R(x^1, x^2 + L, x^3) = \psi_R(x_1, x_2, x_3 + L)$ implies that k_2 and k_3 are quantized according to $k_i = 2\pi n_i/L$ $(i = 2, 3)$. On the other hand, k_2 also has an upper bound since (19.8) shows that the center of the oscillator would be out of the box if k_2 is too large. This condition implies that $k_2/eB < L$, which further gives the maximum value of n_2 to be $(n_2)_{\max} = eBL^2/2\pi$. Note also that the energy eigenvalue does not depend on k_2, thus each energy level consists of $eBL^2/2\pi$ degenerate states. Furthermore, we can also write down explicitly the energy eigenvalue associated with the state labeled by (n_1, n_3):

$$
E = \pm\Big[\Big(\frac{2\pi n_3}{L} - eA\Big)^2 - (n_1 + \tfrac{3}{2})eB\Big]^{1/2}. \tag{19.9}
$$

(e) Now we consider the case with $n_1 = 0$ for simplicity. Then the spectrum reads $E = 2\pi n_3/L - eA$. Suppose that the background potential changes by $\Delta A = 2\pi/eL$. Then it is easy to see that all state with energy marked by n_3 will turn to states with energy marked by n_3-1. Note that each energy eigenvalue is $eBL^2/2\pi$-degenerate, thus the net change of right-handed fermion number is $-eBL^2/2\pi$. Similar analysis shows that the left-handed fermion number get changed by $eBL^2/2\pi$. Therefore the total change is $\Delta N_R - \Delta N_L = -eBL^2/\pi$.

19.2 Weak decay of the pion

(a) In this problem we study the decay of charged pion. So let us work out the amplitude for $\pi^+ \to \ell\nu$, with the effective four-fermion interaction

$$\Delta\mathcal{L} = \frac{4G_F}{\sqrt{2}}(\bar{\ell}_L\gamma^\mu\nu_L)(\bar{u}_L\gamma_\mu d_L) + \text{h.c.} \tag{19.10}$$

and the relation

$$\langle 0|j^{\mu 5a}(x)|\pi^b(p)\rangle = -\mathrm{i}p^\mu f_\pi \delta^{ab} e^{-\mathrm{i}p\cdot x} \tag{19.11}$$

as inputs. Firstly we recall that

$$j^{\mu a} = \bar{Q}_L\gamma^\mu\tau^a Q_L + \bar{Q}_R\gamma^\mu\tau^a Q_R, \tag{19.12a}$$

$$j^{\mu 5a} = -\bar{Q}_L^{\gamma}\mu\tau^a Q_L + \bar{Q}_R\gamma^\mu\tau^a Q_R, \tag{19.12b}$$

where $Q_L = (u_L, d_L)^T$ and $Q_R = (u_R, d_R)^T$. Thus,

$$\frac{1}{2}(j^{\mu 1} + \mathrm{i}j^{\mu 2} - j^{\mu 51} - \mathrm{i}j^{\mu 52}) = \bar{Q}_L\gamma^\mu(\tau^1 + \mathrm{i}\tau^2)Q_L = \bar{u}_L\gamma^\mu d_L. \tag{19.13}$$

Then we find the decay amplitude $\mathcal{M}\big(\pi^+(p) \to \ell^+(k)\nu(q)\big)$ to be

$$\mathrm{i}\mathcal{M} = \frac{4\mathrm{i}G_F}{\sqrt{2}}\bar{u}(q)\gamma^\mu\Big(\frac{1-\gamma_5}{2}\Big)v(k)\cdot\frac{1}{\sqrt{2}}f_\pi\mathrm{i}p_\mu = -G_F f_\pi \bar{u}(q)\not{p}(1-\gamma_5)v(k). \tag{19.14}$$

(b) Now let us calculate the decay rate of the charged pion. We note that the amplitude above can be further simplified to

$$\mathrm{i}\mathcal{M} = -G_F f_\pi \bar{u}(q)(\not{q} + \not{k})(1-\gamma_5)v(k) = -G_F f_\pi m_\ell \bar{u}(q)(1+\gamma_5)v(k). \tag{19.15}$$

Therefore we have

$$\sum|\mathcal{M}|^2 = G_F^2 f_\pi^2 m_\ell^2 \,\mathrm{tr}\,\big(\not{q}(1+\gamma_5)(\not{k} - m_\ell)(1-\gamma_5)\big) = 8G_F^2 f_\pi^2 m_\ell^2 q\cdot k, \tag{19.16}$$

where the summation goes over all final spins. We choose the momenta to be

$$p = (m_\pi, 0, 0, 0), \qquad k = (E_k, 0, 0, k), \qquad q = (E_q, 0, 0, -k). \tag{19.17}$$

Then the kinematics can be easily worked out to be

$$E_k = \frac{m_\pi^2 + m_\ell^2}{2m_\pi^2}, \qquad E_q = k = \frac{m_\pi^2 - m_\ell^2}{2m_\pi} \tag{19.18}$$

The decay rate then follows straightforwardly,

$$\begin{aligned}\Gamma &= \frac{1}{2m_\pi}\int\frac{\mathrm{d}\Omega}{16\pi^2}\frac{k^2}{E_kE_q}\Big(\frac{k}{E_k} + \frac{k}{E_q}\Big)^{-1}\cdot 8G_F^2 f_\pi^2 m_\ell^2(q\cdot k)\\ &= \frac{G_F^2 f_\pi^2}{4\pi m_\pi}\Big(\frac{m_\ell}{m_\pi}\Big)^2(m_\pi^2 - m_\ell^2)^2,\end{aligned} \tag{19.19}$$

and we have the ratio between two decay channels,

$$\frac{\Gamma(\pi^+ \to e^+\nu_e)}{\Gamma(\pi^+ \to \mu^+\nu_\mu)} = \frac{m_e^2(m_\pi^2 - m_e^2)^2}{m_\mu^2(m_\pi^2 - m_\mu^2)^2} \simeq 10^{-4}. \tag{19.20}$$

Thus to determine the pion decay constant f_π, we can consider the channel $\mu^+\nu_\mu$ only as a good approximation. With the lifetime of charged pion $\tau_\pi = 2.6 \times 10^{-8}$sec as well as m_π and m_μ, we find that

$$f_\pi = \sqrt{\frac{4\pi m_\pi}{G_F^2 \tau_\pi}}\Big(\frac{m_\pi}{m_\mu}\Big)(m_\pi^2 - m_\mu^2)^{-1} \simeq 90.6\text{MeV}. \tag{19.21}$$

19.3 Computation of anomaly coefficients

(a) By definition, $\mathcal{A}^{abc} = \operatorname{tr}[t^a, \{t^b, t^c\}]$. Then for the product representation $r_1 \times r_2$, we have

$$\begin{aligned}
\mathcal{A}^{abc}(r_1 \times r_2) &= \operatorname{tr}_{r_1\times r_2}\Big[t^a \otimes 1 + 1 \otimes t^a, \big\{t^b \otimes 1 + 1 \otimes t^b, t^c \otimes 1 + 1 \otimes t^c\big\}\Big] \\
&= \operatorname{tr}_{r_1\times r_2}\Big[t^a \otimes 1 + 1 \otimes t^a, \{t^b, t^c\} \otimes 1 + t^b \otimes t^c + t^c \otimes t^b + 1 \otimes \{t^b, t^c\}\Big] \\
&= \operatorname{tr}_{r_1\times r_2}\Big([t^a\{t^b, t^c\}] \otimes 1 + [t^a, t^b] \otimes t^c + [t^a, t^c] \otimes t^b \\
&\quad + t^b \otimes [t^a, t^c] + t^b \otimes [t^a, t^c] + 1 \otimes [t^a, \{t^b, t^c\}]\Big) \\
&= \operatorname{tr}_{r_1}[t^a, \{t^b, t^c\}] \operatorname{tr}_{r_2}(1) + \operatorname{tr}_{r_2}[t^a, \{t^b, t^c\}] \operatorname{tr}_{r_1}(1) \\
&= \mathcal{A}^{abc}(r_1) d(r_2) + \mathcal{A}^{abc}(r_2) d(r_1).
\end{aligned} \tag{19.22}$$

On the other hand, as we decompose the representation $r_1 \times r_2$ into a direct product of irreducible representations $\sum_i r_i$, we have

$$\begin{aligned}
\mathcal{A}^{abc}\Big(\sum_i r_i\Big) &= \operatorname{tr}_{\Sigma r}\Big[\sum_i t_i^a, \Big\{\sum_j t_j^b, \sum_k t_k^c\Big\}\Big] = \operatorname{tr}_{\Sigma r}\Big(\sum_i \sum_j \sum_k [t_i^a, \{t_j^b, t_k^c\}]\Big) \\
&= \sum_i \operatorname{tr}_{r_i}[t_i^a, \{t_i^b, t_i^c\}] = \sum_i \mathcal{A}^{abc}(r_i)
\end{aligned} \tag{19.23}$$

Note that $\mathcal{A}^{abc}(r) = \frac{1}{2}A(r)d^{abc}$ where d^{abc} is the unique symmetric gauge invariant. Then equating the two expressions above, we get

$$d(r_2)A(r_1) + d(r_1)A(r_2) = \sum_i A(r_i). \tag{19.24}$$

(b) In this part we show that the representation $(\mathbf{3} \times \mathbf{3})_a$ of $SU(3)$ is equivalent to $\bar{\mathbf{3}}$. Let ψ_i be the base vectors of $\mathbf{3}$ representation. Then, a set of base vectors of $(\mathbf{3} \times \mathbf{3})_a$ can be chosen to be $\epsilon_{ijk}\psi_j\psi_k$. From the transformation rule $\psi_i \to U_{ij}\psi_j$, we know that the $(\mathbf{3} \times \mathbf{3})_a$ base vectors transform according to $\epsilon_{ijk}\psi_j\psi_k \to \epsilon_{imn}U_{mj}U_{nk}\psi_j\psi_k$. Now, it is easy to show that $\epsilon_{\ell mn}U_{\ell i}U_{mj}U_{nk}$ is totally antisymmetric, and thus is proportional to ϵ_{ijk}. Let us write $\epsilon_{\ell mn}U_{\ell i}U_{mj}U_{nk} = C\epsilon_{ijk}$, then taking $U = I$ shows that $C = 1$. Now we multiply both sides of this equality by $(U^\dagger)_{ip}$. Since U is unitary, $(U^\dagger)_{ip} = (U^{-1})_{ip}$, so we get $\epsilon_{pmn}U_{mj}U_{nk} = \epsilon_{ijk}(U^\dagger)_{ip}$. That is, the base vector $\epsilon_{ijk}U_jU_k$ transforms according to $\epsilon_{ijk}U_jU_k \to (U^\dagger)_{\ell i}\epsilon_{\ell jk}\psi_j\psi_k = (U^*)_{i\ell}\epsilon_{\ell jk}\psi_j\psi_k$, which is exactly the transformation rule of $\bar{\mathbf{3}}$.

Now from $A(\mathbf{3}) = 1$, it follows that $A(\bar{\mathbf{3}}) = -1$. Therefore $A((\mathbf{3}\times\mathbf{3})_a) = -1$, and by using the equation derived in (a), we have $A((\mathbf{3}\times\mathbf{3})_s) = 6 - (-1) = 7$.

(c) Now we compute the anomaly coefficients for a and s representations of the $SU(N)$ group. As indicated in Peskin & Schroeder, it is enough to consider an $SU(3)$ subgroup of $SU(N)$. Then the fundamental representation $\mathbf{N}$ is decomposed into a direct sum of irreducible representations when restricted to $SU(3)$, that is, $\mathbf{N} = \mathbf{3} + (N-3)\mathbf{1}$. This decomposition is easily justified by considering the upper-left 3×3 block of a matrix in fundamental representation of $SU(N)$. When this block is treated as a transformation of $SU(3)$, the first three components of the vector on which the matrix acts form a fundamental representation vector of $SU(3)$, while the other $(N-3)$ components of the column vector are obviously invariant. With this known, we have,

$$\mathbf{N}\times\mathbf{N} = \big(\mathbf{3} + (N-3)\mathbf{1}\big)\times\big(\mathbf{3} + (N-3)\mathbf{1}\big) = \mathbf{3}\times\mathbf{3} + 2(N-3)\mathbf{3} + (N-3)^2\mathbf{1}. \quad (19.25)$$

On the other hand, we know that $\mathbf{N} = s + a$ while s and a are irreducible. Then we have, by (a), $2N\cdot A(\mathbf{N}) = A(s) + A(a)$. But we already know that $A(\mathbf{N}) = 1$. Thus $A(s) + A(a) = 2N$. Now, to compute $A(a)$, we make use of the $SU(3)$ restriction,

$$(\mathbf{N}\times\mathbf{N})_a = (\mathbf{3}\times\mathbf{3})_a + (N-3)\mathbf{3} + \tfrac{1}{2}(N-3)(N-4)\mathbf{1}. \quad (19.26)$$

Then,

$$A(a) = A\big((\mathbf{3}\times\mathbf{3})_a\big) + (N-3)A(\mathbf{3}) = A(\bar{\mathbf{3}}) + (N-3)A(\mathbf{3}) = N-4, \quad (19.27)$$

and $A(s) = 2N - A(a) = N + 4$.

Now consider totally antisymmetric rank-j tensor representation. Again we decompose the fundamental representation as $\mathbf{N} = \mathbf{3} + (n-3)\mathbf{1}$. Then the rank-$j$ totally antisymmetric tensor can be decomposed as

$$\begin{aligned}(\mathbf{N}^j)_a =\ & \frac{(N-j)\cdots(N-j+1)}{(j-3)!}(\mathbf{3}\times\mathbf{3}\times\mathbf{3})_a + \frac{(N-3)\cdots(N-j)}{(j-2)!}(\mathbf{3}\times\mathbf{3})_a \\ & + \frac{(N-3)\cdots(N-j-1)}{(j-1)!}\mathbf{3} + \mathbf{1}\text{'s}.\end{aligned} \quad (19.28)$$

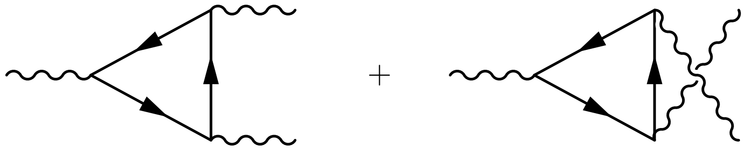
Therefore,

$$\begin{aligned}A(a) &= -\frac{(N-3)\cdots(N-j)}{(j-2)!} + \frac{(N-3)\cdots(N-j-1)}{(j-1)!} \\ &= \frac{(N-3)\cdots(N-j)(N-2j)}{(j-1)!}.\end{aligned} \quad (19.29)$$

19.4 Large fermion mass limits

In this problem we study the chiral anomaly and the trace anomaly in triangle diagrams with Pauli-Villars regularization.

(a) and (c) Firstly we evaluate the expectation value of the divergence of the chiral current $j^{\mu 5}$ between the vacuum and the two-photon state, namely the matrix element $\langle p,k|j^{\mu5}|0\rangle$. This matrix element receives contributions at 1-loop level from the following two diagrams:



In momentum space, the divergence of the first diagram reads

$$\begin{aligned}\mathrm{i}q_\mu\mathcal{M}_1^{\mu\nu\lambda} = (-1)(-\mathrm{i}e)^2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\bigg\{&\,\mathrm{tr}\bigg[\not{q}\gamma^5\frac{\mathrm{i}}{\not{\ell}-\not{k}}\gamma^\lambda\frac{\mathrm{i}}{\not{\ell}}\gamma^\nu\frac{\mathrm{i}}{\not{\ell}+\not{p}}\bigg]\\ &-\mathrm{tr}\bigg[\not{q}\gamma^5\frac{\mathrm{i}}{\not{\ell}-\not{k}-M}\gamma^\lambda\frac{\mathrm{i}}{\not{\ell}-M}\gamma^\nu\frac{\mathrm{i}}{\not{\ell}+\not{p}-M}\bigg]\bigg\}\end{aligned}\tag{19.30}$$

The integral is finite, thus we are allowed to shift the integral variable. For the first trace and the second trace above, we rewrite the $\not{q}\gamma^5$ factors, respectively, as follows,

$$\begin{aligned}&\not{q}\gamma^\mu = (\not{\ell}+\not{p}-\not{\ell}+\not{k})\gamma^\mu = (\not{\ell}+\not{p})\gamma^5+\gamma^5(\not{\ell}-\not{k}),\\ &\not{q}\gamma^\mu = (\not{\ell}+\not{p}-M-\not{\ell}+\not{k}+M)\gamma^\mu = (\not{\ell}+\not{p}-M)\gamma^5+\gamma^5(\not{\ell}-\not{k}-M)+2M\gamma^5.\end{aligned}$$

Then, the loop integral becomes

$$\begin{aligned}\mathrm{i}q_\mu\mathcal{M}_1^{\mu\nu\lambda} = e^2\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\bigg\{&\,\mathrm{tr}\bigg[\gamma^5\frac{1}{\not{\ell}-\not{k}}\gamma^\lambda\frac{1}{\not{\ell}}\gamma^\nu+\gamma^5\gamma^\lambda\frac{1}{\not{\ell}}\gamma^\nu\frac{1}{\not{\ell}+\not{p}}\bigg]\\ &-\mathrm{tr}\bigg[\gamma^5\frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda\frac{1}{\not{\ell}-M}\gamma^\nu+\gamma^5\gamma^\lambda\frac{1}{\not{\ell}-M}\gamma^\nu\frac{1}{\not{\ell}+\not{p}-M}\bigg]\\ &+2M\,\mathrm{tr}\bigg[\gamma^5\frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda\frac{1}{\not{\ell}-M}\gamma^\nu\frac{1}{\not{\ell}+\not{p}-M}\bigg]\bigg\}\end{aligned}\tag{19.31}$$

In the expression above, the first and the second lines are canceled by the corresponding terms from the second diagram with $(k,\lambda\leftrightarrow p,\nu)$, while the third line is doubled. Therefore the sum of two diagrams gives

$$\begin{aligned}\mathrm{i}q_\mu\mathcal{M}^{\mu\nu\lambda} &= 4e^2M\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\,\mathrm{tr}\bigg[\gamma^5\frac{1}{\not{\ell}-\not{k}-M}\gamma^\lambda\frac{1}{\not{\ell}+M}\gamma^\nu\frac{1}{\not{\ell}+\not{p}-M}\bigg]\\ &= 4e^2M\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2N_1}{\big[(\ell-xk+yp)^2-\Delta\big]^3}\\ &= -\frac{4\mathrm{i}e^2MN_1}{(4\pi)^2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1}{\Delta}\end{aligned}\tag{19.32}$$

with

$$\begin{aligned} N_1 =&\ \mathrm{tr}\left[\gamma^5(\not{\ell} - \not{k} + M)\gamma^\lambda(\not{\ell} - M)\gamma^\nu(\not{\ell} + \not{p} + M)\right] = -4\mathrm{i}M\epsilon^{\alpha\beta\lambda\nu}k_\alpha p_\beta, \\ \Delta =&\ M^2 - x(1-x)k^2 - y(1-y)p^2 - 2xyk\cdot p. \end{aligned}$$

Then the integral can be carried out directly in the $M^2 \to \infty$ limit, to be

$$\mathrm{i}q_\mu \mathcal{M}^{\mu\nu\lambda} = -\frac{e^2}{2\pi^2}\epsilon^{\alpha\beta\lambda\nu}k_\alpha p_\beta, \tag{19.33}$$

as expected.

(b) and (d) For scale anomaly, the diagrams are the same. Now the relevant matrix element is given by $\langle p,k|M\bar{\psi}\psi|0\rangle$. Then the first diagram reads

$$\begin{aligned} \mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \mathrm{i}e^2M\int\frac{\mathrm{d}^4\ell}{(2\pi)^4}\bigg\{&\bigg[\frac{1}{\not{\ell}-\not{k}}\not{\epsilon}^*(k)\frac{1}{\not{k}}\not{\epsilon}^*(p)\frac{1}{\not{\ell}+\not{p}}\bigg] \\ &- \mathrm{tr}\bigg[\frac{1}{\not{\ell}-\not{k}-M}\not{\epsilon}^*(k)\frac{1}{\not{\ell}-M}\not{\epsilon}^*(p)\frac{1}{\not{\ell}+\not{p}-M}\bigg]\bigg\} \end{aligned} \tag{19.34}$$

The first trace vanishes upon regularization, then,

$$\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = -\,\mathrm{i}e^2M\int\frac{\mathrm{d}^4\ell'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2N_2}{(\ell'^2-\Delta)^3}, \tag{19.35}$$

where $\ell' = \ell - xk + yp$, $\Delta = M^2 - 2xyk\cdot p$, and the trace in the numerator is

$$\begin{aligned} N_2 =&\ \mathrm{tr}\left[(\not{\ell} - k + M)\not{\epsilon}^*(k)(\not{\ell}+M)\not{\epsilon}^*(p)(\not{\ell}+\not{p}+M)\right] \\ =&\ 4M\big[M^2\epsilon^*(k)\cdot\epsilon^*(p) + \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big) - \big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\big(k\cdot p\big) \\ &\quad + 4\big(\epsilon^*(k)\cdot\ell\big)\big(\epsilon^*(p)\cdot\ell\big) - \big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\ell^2\big] \\ =&\ 4M\big[M^2\epsilon^*(k)\cdot\epsilon^*(p) + (1-4xy)\big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big) \\ &\quad - (1-2xy)\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) + \big(\tfrac{4}{d}-1\big)\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)\ell'^2\big], \end{aligned}$$

where we used the transverse condition $k\cdot\epsilon^*(k) = p\cdot\epsilon^*(p) = 0$, and in the last equality, the substitution $\ell'^\mu\ell'^\nu \to \frac{1}{d}g^{\mu\nu}\ell'^2$. We also dropped all terms linear in ℓ' in the last equality. The integral is then divergent, and we regularize it by dimensional regularization. Then after carrying out the loop integral, we get

$$\begin{aligned} \mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) =&\ \frac{e^2}{4\pi^2}\Big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\Big] \\ &\times\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{(1-4xy)M^2}{M^2-2xyk\cdot p} \end{aligned} \tag{19.36}$$

Then, taking $M^2\to\infty$ limit, we find

$$\mathrm{i}\mathcal{M}_1^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \frac{e^2}{12\pi^2}\Big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\Big] \tag{19.37}$$

The second diagram is obtained, again, by the exchange $(k,\lambda \leftrightarrow p,\nu)$, which gives the identical result. Therefore we finally get

$$\mathrm{i}\mathcal{M}^{\mu\nu\lambda}\epsilon^*_\nu(p)\epsilon^*_\lambda(k) = \frac{e^2}{6\pi^2}\Big[\big(\epsilon^*(k)\cdot\epsilon^*(p)\big)(k\cdot p) - \big(\epsilon^*(k)\cdot p\big)\big(\epsilon^*(p)\cdot k\big)\Big]. \tag{19.38}$$

Chapter 20

Gauge Theories with Spontaneous Symmetry Breaking

20.1 Spontaneous breaking of $SU(5)$

We consider two patterns of spontaneous breaking of $SU(5)$ gauge symmetry, with an adjoint-representation scalar field Φ picking up vacuum expectation values

$$\langle\Phi\rangle = A\text{diag}(1,1,1,1,-4), \qquad \langle\Phi\rangle = B\text{diag}(2,2,2,-3,-3), \tag{20.1}$$

respectively. The kinetic term of the scalar field in the Lagrangian is

$$\mathcal{L}_{\text{kin.}} = \text{tr}\Big((\text{D}_\mu\Phi)^\dagger(\text{D}^\mu\Phi)\Big) = \text{tr}\Big(\big(\partial_\mu\Phi + g[A_\mu,\Phi]\big)^\dagger\big(\partial^\mu\Phi + g[A^\mu,\Phi]\big)\Big). \tag{20.2}$$

Then the mass term of gauge bosons after symmetry breaking is given by

$$\Delta\mathcal{L} = g^2\,\text{tr}\Big([A_\mu,\Phi]^\dagger[A^\mu,\Phi]\Big) = -g^2A_\mu^aA^{\mu b}\,\text{tr}\Big([T^a,\langle\Phi\rangle][T^b,\langle\Phi\rangle]\Big). \tag{20.3}$$

To analyze the gauge bosons' spectrum, we note that there are 24 independent generators for $SU(5)$ group, each of which can be represented as a 5×5 traceless hermitian matrix. Then, for the first choice of $\langle\Phi\rangle = \text{diag}(1,1,1,1,-4)$, we see that for the generators of the form

$$T = \begin{pmatrix} T^{(4)} & \\ & 0 \end{pmatrix} \qquad \text{and} \qquad T = \tfrac{1}{2\sqrt{10}}\text{diag}(1,1,1,1,-4),$$

where $T^{(4)}$ is a 4×4 matrix being any generator of $SU(4)$ group, the commutators vanish. That is, a subgroup $SU(4)\times U(1)$ remains unbroken in this case. Then, for the rest of the generators, namely

$$\frac{1}{2}\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0&0&0&0&\text{i}\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ -\text{i}&0&0&0&0 \end{pmatrix}, \qquad \frac{1}{2}\begin{pmatrix} 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&1&0&0&0 \end{pmatrix},$$

etc, it is easy to calculate the commutators to get the trace equal to $-25A^2/2$. Thus the corresponding components of gauge bosons acquire mass $M_A = 5gA$. In the same way, we can also analyze the case of $\langle\Phi\rangle = \text{diag}(2,2,2,-3,-3)$. This time the unbroken subgroup is $SU(3)\times SU(2)\times U(1)$, and the remaining 12 components of gauge bosons acquire a mass equal to $M_A = 5gB$, as can be found by evaluating the corresponding commutators.

20.2 Decay modes of the W and Z bosons

(a) The relevant interaction term in the Lagrangian reads

$$\Delta\mathcal{L} = \tfrac{1}{\sqrt{2}}gW_\mu^+\Big(\sum_i \bar{\nu}_{iL}\gamma^\mu e_{iL} + \sum_{j,c}\bar{u}^c_{jL}\gamma^\mu d^c_{jL}\Big), \tag{20.4}$$

where the sum on i goes over all three generations of leptons, the sum on j goes over the first two generations of quarks, since $m_t > m_W$, and the sum on c is due to 3 colors.

Now consider the decay of W^+ boson. The amplitude of the decay into a pair of fermions is

$$\text{i}\mathcal{M} = \frac{\text{i}g}{\sqrt{2}}\epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)v(p_2), \tag{20.5}$$

where ϵ_μ is the polarization vector for W_μ^+, and the labels for momenta are shown in Fig. 20.1. Thus the squared amplitude with initial polarizations averaged is

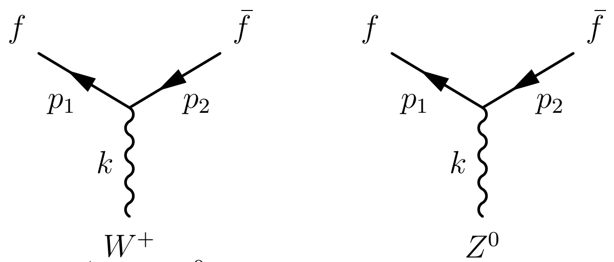


Figure 20.1: The decay of W^+ and Z^0 into fermion-antifermion pairs. All initial momenta go inward and all final momenta go outward.

$$\begin{aligned}\frac{1}{3}\sum_{\text{spin}}|\text{i}\mathcal{M}|^2 &= \frac{g^2}{6}\Big(-g_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\Big)\,\text{tr}\Big[\not{p}_2\gamma^\mu\Big(\frac{1-\gamma^5}{2}\Big)\not{p}_1\gamma^\nu\Big(\frac{1-\gamma^5}{2}\Big)\Big]\\ &= \frac{g^2}{3}\Big(p_1\cdot p_2 + 2\frac{k\cdot p_1 k\cdot p_2}{m_W^2}\Big).\end{aligned} \tag{20.6}$$

The momenta in the center-of-mass frame can be taken to be

$$k = (m_W,0,0,0), \qquad p_1 = (p,0,0,p), \qquad p_2 = (p,0,0,-p), \tag{20.7}$$

and energy conservation requires that $p = m_W/2$. Thus we get

$$\frac{1}{3}\sum_{\text{spin}}|\text{i}\mathcal{M}|^2 = \frac{1}{3}g^2 m_W^2, \tag{20.8}$$

and the decay rate

$$\int d\Gamma = \frac{1}{2m_W}\int \frac{d^3p_1 d^3p_2}{(2\pi)^6 2E_1 2E_2}\left(\frac{1}{3}g^2 m_W^2\right)(2\pi)^4\delta^{(4)}(k-p_1-p_2) = \frac{\alpha m_W}{12\sin^2\theta_w}, \qquad (20.9)$$

where we have used $g = e/\sin\theta_w$ and $\alpha = e^2/4\pi$. For each quark final state we multiply the result by a QCD correction factor $\left(1+\frac{\alpha_s}{\pi}\right)$. Then, taking account of 3 generations of leptons and 2 generations of quarks with 3 colors, we get the partial decay rate of W^+ into fermions,

$$\Gamma(W^+ \to e_i^+\nu_i) = \frac{\alpha m_W}{12\sin^2\theta_w} \simeq 0.23\text{GeV}; \qquad (20.10)$$

$$\Gamma(W^+ \to u_j\bar{d}_j) = \frac{\alpha m_W}{4\sin^2\theta_w}\left(1+\frac{\alpha_s}{\pi}\right) \simeq 0.70\text{GeV}; \qquad (20.11)$$

$$\Gamma(W^+ \to \text{fermions}) = \frac{\alpha m_W}{12\sin^2\theta_w}\left(9+6\frac{\alpha_s}{\pi}\right) \simeq 2.08\text{GeV}. \qquad (20.12)$$

and also the branching ratios $\text{BR}(W^+ \to e_i^+\nu_i) = 0.11\%$, and $\text{BR}(W^+ \to u_j\bar{d}_j) = 0.34\%$. Note that the fine structure constant at m_W is $\alpha(m_W) \simeq 1/129$.

(b) In the same way, we can also calculate the decay rate of $Z \to$fermions. The relevant term in the Lagrangian is

$$\Delta\mathcal{L} = \frac{g}{\cos\theta_w} Z_\mu \sum_i \bar{f}_i\gamma^\mu\left(I_i^3 - \sin^2\theta_w Q_i\right)f_i, \qquad (20.13)$$

where the sum goes over all left- and right-handed fermions, including 3 generations of leptons, and the first two generations of quarks with 3 colors, while I_i^3 and Q_i are associated 3-component of the weak isospin and the electric charge, respectively.

Then we can write down the amplitudes of the decay of Z^0 into a pair of fermions $f\bar{f}$ with specific I^3 and Q, as illustrated in Fig. 20.3,

$$\begin{aligned} i\mathcal{M} &= \frac{ig}{\cos\theta_w}\epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\left[\left(I^3 - \sin^2\theta_w Q\right)\left(\frac{1-\gamma^5}{2}\right) - \sin^2\theta_w Q\left(\frac{1+\gamma^5}{2}\right)\right]v(p_2) \\ &= \frac{ig}{\cos\theta_w}\epsilon_\mu(k)\bar{u}(p_1)\gamma^\mu\left[I^3\left(\frac{1-\gamma^5}{2}\right) - \sin^2\theta_w Q\right]v(p_2), \end{aligned} \qquad (20.14)$$

the squared matrix elements,

$$\begin{aligned} \frac{1}{3}\sum_{\text{spin}}|i\mathcal{M}|^2 &= \frac{g^2}{3\cos^2\theta_w}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2}\right) \\ &\qquad\times \text{tr}\left[\not{p}_2\gamma^\mu\left(\tfrac{1}{2}I^3(1-\gamma^5) - \sin^2\theta_w Q\right)\not{p}_1\gamma^\nu\left(\tfrac{1}{2}I^3(1-\gamma^5) - \sin^2\theta_w Q\right)\right] \\ &= \frac{4g^2}{3\cos\theta_w^2}\left[\left(\tfrac{1}{2}I^3 - \sin^2\theta_w Q\right)^2 + \left(\tfrac{1}{2}I^3\right)^2\right]\left(p_1\cdot p_2 + \frac{2k\cdot p_1 k\cdot p_2}{m_Z^2}\right) \\ &= \frac{4g^2 m_Z^2}{3\cos\theta_w^2}\left[\left(\tfrac{1}{2}I^3 - \sin^2\theta_w Q\right)^2 + \left(\tfrac{1}{2}I^3\right)^2\right], \end{aligned} \qquad (20.15)$$

and the partial decay rate,

$$\Gamma(Z^0 \to f\bar{f}) = \frac{\alpha m_Z}{3\sin^2\theta_w \cos^2\theta_w}\Big[\big(\tfrac{1}{2}I^3 - \sin^2\theta_w Q\big)^2 + \big(\tfrac{1}{2}I^3\big)^2\Big]. \tag{20.16}$$

We should also multiply the result for quarks by the QCD factor $\big(1+\frac{\alpha_s}{\pi}\big)$. Now we list the numerical results of partial width and the branching ratios for various decay products as follows.

$f\bar{f}$	$\Gamma(f\bar{f})$/GeV	BR($f\bar{f}$)
$\nu_e\bar{\nu}_e$, $\nu_\mu\bar{\nu}_\mu$, $\nu_\tau\bar{\nu}_\tau$	0.17	6.7%
e^-e^+, $\mu^-\mu^+$, $\tau^-\tau^+$	0.08	3.4%
$u\bar{u}$, $c\bar{c}$	0.30	11.9%
$d\bar{d}$, $s\bar{s}$, $b\bar{b}$	0.39	15.4%
All fermions	2.51	100%

20.3 e^+e^- →hadrons with photon-Z^0 interference

(a) It is easier to work with amplitudes between initial and final fermions with definite chirality. In this case the relevant amplitude is given by*

$$\mathrm{i}\mathcal{M} = (\mathrm{i}e)^2\bar{v}(k_2)\gamma^\mu u(k_1)\frac{-\mathrm{i}}{q^2}\bar{u}(p_1)\gamma_\mu Q_f v(p_2)\bigg[-Q_f + \frac{(I_e^3+s_w^2)(I_f^3-s_w^2Q_f)}{s_w^2c_w^2}\frac{q^2}{q^2-m_Z^2}\bigg], \tag{20.17}$$

where $I_e^3 = -1/2$ or 0 when the initial electron is left-handed or right-handed, so as I_f^3 to the final fermion. The momenta is labeled as shown in Fig. 20.2. Then we can find associated

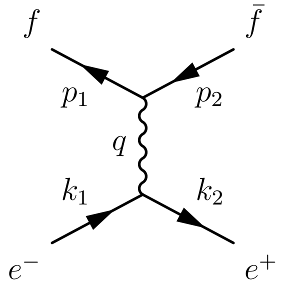


Figure 20.2: The process of $e^+e^- \to f\bar{f}$ via the exchange of a photon/Z^0 in s-channel. The directions of k_i's and p_i's are inward and outward, respectively.

differential cross section to be

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_R^+e_L^- \to \bar{f}_R f_L) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2F_{LL}(f), \tag{20.18a}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_R^+e_L^- \to \bar{f}_L f_R) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2F_{LR}(f), \tag{20.18b}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_L^+e_R^- \to \bar{f}_R f_L) = \frac{\pi\alpha^2}{2s}(1-\cos\theta)^2F_{RL}(f), \tag{20.18c}$$

*In this problem we simplify the notation by $s_w \equiv \sin\theta_w$ and $c_w \equiv \cos\theta_w$.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}(e_L^+e_R^- \to \bar{f}_L f_R) = \frac{\pi\alpha^2}{2s}(1+\cos\theta)^2 F_{RR}(f), \tag{20.18d}$$

in which α is the fine structure constant, $s = q^2$ is the center-of-mass energy, and the F factors are defined as follows:

$$F_{LL}(f) = \left| Q_f + \frac{(\frac{1}{2}-s_w^2)(I_f^3 - s_w^2 Q_f)}{s_w^2 c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.19}$$

$$F_{LR}(f) = \left| Q_f - \frac{(\frac{1}{2}-s_w^2)Q_f}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.20}$$

$$F_{RL}(f) = \left| Q_f - \frac{(I_f^3 - s_w^2 Q_f)}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.21}$$

$$F_{RR}(f) = \left| Q_f + \frac{s_w^2 Q_f}{c_w^2} \frac{s}{s - m_Z^2 + \mathrm{i}m_Z\Gamma_Z} \right|^2, \tag{20.22}$$

where we have added the correction from resonance by using the Breit-Wigner formula. Summing up the four expressions in (20.18), averaging the initial spins, and integrating over the angle θ, we get finally the unpolarized cross section

$$\sigma(f\bar{f}) = \frac{\pi\alpha^2}{3s}\Big[F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + F_{RR}(f)\Big]. \tag{20.23}$$

When the final state particle f is a quark, one should multiply the result by $3\big(1+\frac{\alpha_s}{\pi}\big)$ where 3 is the color factor, and $\big(1+\frac{\alpha_s}{\pi}\big)$ is the 1-loop QCD correction.

For the final fermion being muon ($I_f^3 = -1/2$, $Q_f = -1$), up quark ($I_f^3 = 1/2$, $Q = 2/3$), and down quark ($I_f^3 = -1/2$, $Q_f = -1/3$), we plot the corresponding cross section as a function of center-of-mass energy $E_{\text{CM}} = \sqrt{s}$ in Fig. 20.3.

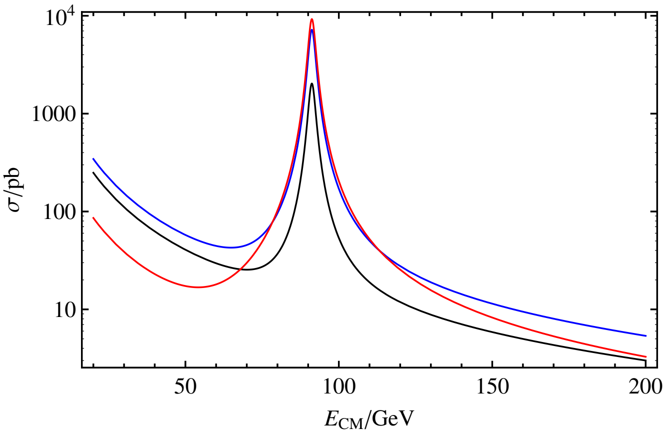


Figure 20.3: The cross section $\sigma(e^+e^- \to f\bar{f})$ as a function of center-of-mass energy E_{CM}. The black, blue, and red curves correspond to $f\bar{f} = \mu^-\mu^+$, $u\bar{u}$ and $d\bar{d}$, respectively.

(b) Now we calculate the forward-backward asymmetry A_{FB}^f, defined to be

$$A_{FB}^f = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{\big(\int_0^1 - \int_{-1}^0\big)\mathrm{d}\cos\theta(\mathrm{d}\sigma/\mathrm{d}\cos\theta)}{\big(\int_0^1 + \int_{-1}^0\big)\mathrm{d}\cos\theta(\mathrm{d}\sigma/\mathrm{d}\cos\theta)}. \tag{20.24}$$

Then from (20.18), we find

$$\sigma_F = \frac{\pi\alpha^2}{24s}\Big[7F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + 7F_{RR}(f)\Big], \tag{20.25}$$

$$\sigma_B = \frac{\pi\alpha^2}{24s}\Big[F_{LL}(f) + 7F_{LR}(f) + 7F_{RL}(f) + F_{RR}(f)\Big]. \tag{20.26}$$

Thus

$$A_{FB}^f = \frac{3}{4}\cdot\frac{F_{LL}(f) - F_{LR}(f) - F_{RL}(f) + F_{RR}(f)}{F_{LL}(f) + F_{LR}(f) + F_{RL}(f) + F_{RR}(f)}. \tag{20.27}$$

Again, we plot A_{FB}^f, as a function of $E_{\rm CM}$, for $f = \mu^-, u, d$, in Fig. 20.4.

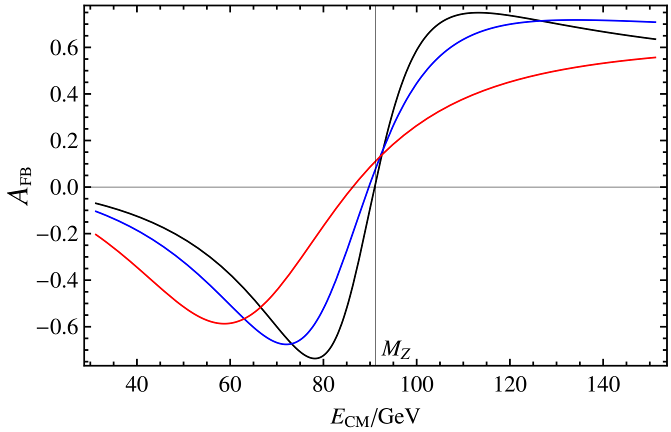


Figure 20.4: The forward-backward asymmetry A_{FB}^f as a function of center-of-mass energy $E_{\rm CM}$. The black, blue, and red curves correspond to $f\bar{f} = \mu^-\mu^+$, $u\bar{u}$ and $d\bar{d}$, respectively.

(c) Recall the definition of F's, we find, on the Z^0 resonance ($s = m_Z$),

$$F_{LL}(f) \simeq \left[\frac{\big(\frac{1}{2}-s_w^2\big)\big(I_f^3 - s_w^2 Q_f\big)}{s_w^2 c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2, \qquad F_{LR}(f) \simeq \left[\frac{\big(\frac{1}{2}-s_w^2\big)Q_f}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2,$$

$$F_{RL}(f) \simeq \left[\frac{\big(I_f^3 - s_w^2 Q_f\big)}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2, \qquad F_{RR}(f) \simeq \left[\frac{s_w^2 Q_f}{c_w^2}\frac{m_Z}{\Gamma_Z}\right]^2,$$

therefore,

$$A_{FB}^f = \frac{3}{4}\cdot\frac{\Big[\big(\frac{1}{2}-s_w^2\big)^2 - s_w^4\Big]\Big[\big(I_f^3 - s_w^2 Q_f\big)^2 - (s_w^2 Q_f)^2\Big]}{\Big[\big(\frac{1}{2}-s_w^2\big)^2 + s_w^4\Big]\Big[\big(I_f^3 - s_w^2 Q_f\big)^2 + (s_w^2 Q_f)^2\Big]} = \frac{3}{4}A_{LR}^e A_{LR}^f. \tag{20.28}$$

(d)

$$\begin{aligned}
\sigma_{\rm peak} &= \frac{\pi\alpha^2}{3m_Z^2}\cdot\frac{1}{s_w^4 c_w^4}\cdot\frac{m_Z^2}{\Gamma_Z^2}\Big[\big(\tfrac{1}{2}-s_w^2\big)^2 + s_w^4\Big]\Big[\big(I_f^3 - s_w^2 Q_f\big)^2 + (s_w^2 Q_f)^2\Big] \\
&= \frac{12\pi}{m_Z^2\Gamma_Z^2}\left(\frac{\alpha m_Z}{6s_w^2 c_w^2}\Big[\big(\tfrac{1}{2}-s_w^2\big)^2 + s_w^4\Big]\right)\left(\frac{\alpha m_Z}{6s_w^2 c_w^2}\Big[\big(I_f^3 - s_w^2 Q_f\big)^2 + (s_w^2 Q_f)^2\Big]\right) \\
&= \frac{12\pi}{m_Z^2}\cdot\frac{\Gamma(Z^0\to e^+e^-)\Gamma(Z^0\to f\bar{f})}{\Gamma_Z^2}.
\end{aligned} \tag{20.29}$$

20.4 Neutral-current deep inelastic scattering

(a) In this problem we study the neutral-current deep inelastic scattering. The process is mediated by Z^0 boson. Assuming m_Z is much larger than the energy scale of the scattering process, we can write down the corresponding effective operators, from the neutral-current Feynman rules in electroweak theory,

$$\Delta\mathcal{L} = \frac{g^2}{4m_W^2}(\bar{\nu}\gamma^\mu)P_L\nu\Big[\bar{u}\gamma_\mu\Big(\big(1-\tfrac{4}{3}s_w^2\big)P_L - \tfrac{4}{3}s_w^2P_R\Big)u \\ + \bar{d}\gamma_\mu\Big(\big(1-\tfrac{2}{3}s_w^2\big)P_L - \tfrac{2}{3}s_w^2P_R\Big)d\Big] + \text{h.c.}, \tag{20.30}$$

where $P_L=(1-\gamma^5)/2$ and $P_R=(1+\gamma^5)/2$ are left- and right-handed projectors, respectively. Compare the effective operator with the charged-operator in (17.31) of Peskin & Schroeder, we can write down directly the differential cross section by modifying (17.35) in Peskin & Schroeder properly, as

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu p\to\nu X) = \frac{G_F^2 sx}{4\pi}\Big\{&\Big[\big(1-\tfrac{4}{3}s_w^2\big)^2+\tfrac{16}{9}s_w^4(1-y^2)\Big]f_u(x)\\ &+\Big[\big(1-\tfrac{2}{3}s_w^2\big)^2+\tfrac{4}{9}s_w^4(1-y^2)\Big]f_d(x)\\ &+\Big[\tfrac{16}{9}s_w^4+\big(1-\tfrac{4}{3}s_w^2\big)^2(1-y^2)\Big]f_{\bar{u}}(x)\\ &+\Big[\tfrac{4}{9}s_w^4+\big(1-\tfrac{2}{3}s_w^2\big)^2(1-y^2)\Big]f_{\bar{d}}(x)\Big\},\end{aligned} \tag{20.31}$$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\bar{\nu} p\to\bar{\nu} X) = \frac{G_F^2 sx}{4\pi}\Big\{&\Big[\tfrac{16}{9}s_w^4+\big(1-\tfrac{4}{3}s_w^2\big)^2(1-y^2)\Big]f_u(x)\\ &+\Big[\tfrac{4}{9}s_w^4+\big(1-\tfrac{2}{3}s_w^2\big)^2(1-y^2)\Big]f_d(x)\\ &+\Big[\big(1-\tfrac{4}{3}s_w^2\big)^2+\tfrac{16}{9}s_w^4(1-y^2)\Big]f_{\bar{u}}(x)\\ &+\Big[\big(1-\tfrac{2}{3}s_w^2\big)^2+\tfrac{4}{9}s_w^4(1-y^2)\Big]f_{\bar{d}}(x)\Big\}.\end{aligned} \tag{20.32}$$

(b) For the neutrino scattering from a nucleus A with equal numbers of protons and neutrons, we have $f_u=f_d$ and $f_{\bar{u}}=f_{\bar{d}}$. Then the differential cross sections reads

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\nu A\to\nu X) = \frac{G_F^2 sx}{\pi}\Big\{&\big[\tfrac{1}{2}-s_w^2+\tfrac{5}{9}s_4^4+\tfrac{5}{9}s_4^4(1-y^2)\big]f_u(x)\\ &+\big[\tfrac{5}{9}s_w^4+\big(\tfrac{1}{2}-s_w^2+\tfrac{5}{9}s_w^4\big)(1-y^2)\big]f_{\bar{u}}(x)\Big\},\end{aligned} \tag{20.33}$$

$$\begin{aligned}\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}(\bar{\nu} p\to\bar{\nu} X) = \frac{G_F^2 sx}{\pi}\Big\{&\big[\tfrac{5}{9}s_w^4+\big(\tfrac{1}{2}-s_w^2+\tfrac{5}{9}s_w^4\big)(1-y^2)\big]f_u(x)\\ &+\big[\tfrac{1}{2}-s_w^2+\tfrac{5}{9}s_4^4+\tfrac{5}{9}s_4^4(1-y^2)\big]f_{\bar{u}}(x)\Big\}.\end{aligned} \tag{20.34}$$

Recall that for charged-current neutrino deep inelastic scattering, the differential cross sections are given by (17.35) in Peskin & Schroeder. Thus it is easy to find that

$$R^\nu = \frac{\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y(\nu A\to\nu X)}{\mathrm{d}^\sigma/\mathrm{d}x\mathrm{d}y(\nu A\to\mu^- X)} = \frac{1}{2}-s_w^2+\frac{5}{9}s_w^4\bigg(1+\frac{f_u(x)(1-y^2)+f_{\bar{u}}(x)}{f_u(x)+f_{\bar{u}}(x)(1-y)^2}\bigg), \tag{20.35}$$

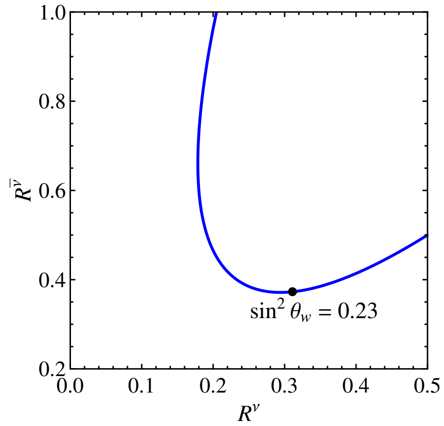


Figure 20.5: Weinberg's nose with $r = 0.4$. See problem 21.4.

$$R^{\bar{u}} = \frac{\mathrm{d}^2\sigma/\mathrm{d}x\mathrm{d}y(\bar{\nu}A \to \bar{\nu}X)}{\mathrm{d}^\sigma/\mathrm{d}x\mathrm{d}y(\bar{\nu}A \to \mu^+X)} = \frac{1}{2} - s_w^2 + \frac{5}{9}s_w^4\left(1 + \frac{f_u(x) + f_{\bar{u}}(x)(1-y)^2}{f_u(x)(1-y^2) + f_{\bar{u}}(x)}\right), \quad (20.36)$$

where

$$\frac{f_u(x)(1-y^2) + f_{\bar{u}}(x)}{f_u(x) + f_{\bar{u}}(x)(1-y)^2} = r. \quad (20.37)$$

(c) The plot "Weinberg's Nose" with $r = 0.4$ is shown in Figure 20.5.

20.5 A model with two Higgs fields

(a) The gauge boson mass matrix comes from the kinetic term of scalar fields,

$$(\mathrm{D}_\mu\phi_1)^\dagger(\mathrm{D}^\mu\phi_1) + (\mathrm{D}_\mu\phi_2)^\dagger(\mathrm{D}^\mu\phi_2),$$

with $\mathrm{D}_\mu\phi_{1,2} = \big(\partial_\mu - \frac{\mathrm{i}}{2}gA_\mu^a\sigma^a - \frac{\mathrm{i}}{2}g'B_\mu\big)\phi_{1,2}$. After $\phi_{1,2}$ acquire the vacuum expectation value $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$, we observe that each of the kinetic terms gives rise to mass terms for gauge bosons similar to the ones in the standard electroweak theory. Thus it is straightforward that the masses of gauge bosons in this model is given by the replacement $v^2 \to v_1^2 + v_2^2$.

(b) The statement that the configuration $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$ is a locally stable minimum, is equivalent to that all particle excitations generated above this solution have positive squared mass m^2. Thus we investigate the mass spectrum of the theory with the vacuum chosen to be $\frac{1}{\sqrt{2}}\binom{0}{v_{1,2}}$. Firstly, we parameterize two scalar doublets as

$$\phi_i = \begin{pmatrix} \pi_i^+, \\ \frac{1}{\sqrt{2}}(v_i + h_i + \mathrm{i}\pi_i^0) \end{pmatrix}, \qquad (i = 1, 2) \quad (20.38)$$

and substitute this parameterization into the potential,

$$\begin{aligned} V = &- \mu_1^2\phi_1^\dagger\phi_1 - \mu_2^2\phi_2^\dagger\phi_2 + \lambda_1(\phi_1^\dagger\phi)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 \\ &+ \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \lambda_5\big((\phi_1^\dagger\phi_2)^2 + \text{h.c.}\big). \end{aligned} \quad (20.39)$$

Then the mass term of various scalar components can be extracted, as follows.

$$
\begin{aligned}
\mathcal{L}_{\text{mass}} =&\ (\lambda_4 + 2\lambda_5)v_1v_2 \begin{pmatrix} \pi_1^- & \pi_2^- \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} \\
&+ 2\lambda_5 v_1 v_2 \begin{pmatrix} \pi_1^0 & \pi_2^0 \end{pmatrix} \begin{pmatrix} v_2/v_1 & -1 \\ -1 & v_1/v_2 \end{pmatrix} \begin{pmatrix} \phi_1^0 \\ \phi_2^0 \end{pmatrix} \\
&- v_1 v_2 \begin{pmatrix} h_1 & h_2 \end{pmatrix} \begin{pmatrix} \lambda_1(v_1/v_2) & \lambda_3+\lambda_4+2\lambda_5 \\ \lambda_3+\lambda_4+2\lambda_5 & \lambda_2(v_2/v_1) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\end{aligned} \tag{20.40}
$$

The eigenvalues of these matrices are easy to be found. For charged components, there is a zero mode corresponding two broken directions in $SU(2)$, and the mass of the other charged scalar is given by $m_c^2 = -(\lambda_4 + 2\lambda_5)(v_1^2 + v_2^2)$. For pseudoscalar components, there is also a zero mode corresponding to the rest one direction of broken $SU(2)$, and the mass of the other pseudoscalar is $m_p^2 = -4\lambda_5(v_1^2 + v_2^2)$. Finally, for neutral scalars, the two mass eigenvalues are given by the roots of following equation,

$$
m_n^2 - (\lambda_1 v_1^2 + \lambda_2 v_2^2)m_n^2 + \big[\lambda_1\lambda_2 - (\lambda_3 + \lambda_4 + 2\lambda_5)^2\big] = 0. \tag{20.41}
$$

Therefore, to make $m_c^2 > 0$, $m_p^2 > 0$ and $m_n^2 > 0$, it is sufficient that

$$
\lambda_4 + 2\lambda_5 < 0, \qquad \lambda_5 < 0, \qquad \lambda_1, \lambda_2 > 0, \qquad \lambda_1\lambda_2 > (\lambda_3 + \lambda_4 + 2\lambda_5)^2. \tag{20.42}
$$

(c) From the mass terms in (b) we can diagonalize the charged scalar mass matrix with the rotation matrix

$$
\begin{pmatrix} \pi^+ \\ \phi^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \pi_1^+ \\ \pi_2^+ \end{pmatrix}, \tag{20.43}
$$

where π^+ is the Goldstone mode and ϕ^+ is a physical charged scalar. Given that ϕ^+ to get the physical mass, it is easy to see that the rotation angle can be chosen to be $\tan\beta = v_2/v_1$.

(d) Assuming that the Yukawa interactions between quarks and scalars take the following form,

$$
\mathcal{L}_m = -\begin{pmatrix} \bar{u}_L & \bar{d}_L \end{pmatrix} \left[\lambda_d \begin{pmatrix} \pi_1^+ \\ \frac{1}{\sqrt{2}}v_1 \end{pmatrix} d_R + \lambda_u \begin{pmatrix} \frac{1}{\sqrt{2}}v_2 \\ \pi^- \end{pmatrix} u_R \right] + \text{h.c.}, \tag{20.44}
$$

where we have suppressed flavor indices and neglected neutral scalar components. We focus on charged component only. Then, with Peskin & Schroeder's notation, we make the replacement $u_L \to U_u u_L$, $d_L \to U_d d_L$, $u_R \to W_u u_R$, and $d_R \to W_d d_R$. Then, together with $\lambda_d = U_d D_d W_d^\dagger$ and $\lambda_u = U_u D_u W_u^\dagger$ where D_d and D_u are diagonal matrix, we have

$$
\begin{aligned}
\mathcal{L}_m =& -\frac{1}{\sqrt{2}}\big(v_1 \bar{d}_L D_d d_R + v_2 \bar{u}_L D_u u_R\big) \\
&- \bar{u} V_{\text{CKM}} D_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger D_u u_R \pi_2^- + \text{h.c.}
\end{aligned} \tag{20.45}
$$

From the first line we see that the diagonal mass matrix for quarks are given by $m_u = (v_1/\sqrt{2})D_u$ and $m_d = (v_2/\sqrt{2})D_d$. We further define $v = \sqrt{v_1^2 + v_2^2}$ and note that $\pi_1^+ = -\phi^+ \sin\beta \cdots$, $\pi_2^+ = \phi^+ \cos\beta + \cdots$, then the Yukawa interactions between charged boson and quarks can be written as

$$\begin{aligned}\mathcal{L}_m \Rightarrow& -\frac{\sqrt{2}}{v_1}\Big(\bar{u}_L V_{\text{CKM}} m_d d_R \pi_1^+ + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \pi_2^-\Big) + \text{h.c.}\\ \Rightarrow& \frac{\sqrt{2}}{v}\Big(\bar{u}_L V_{\text{CKM}} m_d d_R \phi^+ \tan\beta + \bar{d}_L V_{\text{CKM}}^\dagger m_u u_R \phi^- \cot\beta\Big) + \text{h.c.}.\end{aligned} \tag{20.46}$$

Chapter 21

Quantization of Spontaneously Broken Gauge Theories

21.1 Weak-interaction contributions to the muon $g-2$

In this problem we study the weak-interaction corrections to the muon's anomalous magnetic moment (AMM). The relevant contributions come from the W-neutrino loop and Z-muon loop, together with the diagrams with the gauge bosons replaced with the corresponding Goldstone bosons. Here we will evaluate the W-neutrino loop diagram with Feynman-'t Hooft gauge and general R_ξ gauge in part (a) and part (b) respectively, and Z-muon diagram in part (c).

(a) Now we come to the W-neutrino loop diagram and the corresponding Goldstone boson diagrams, shown in Fig. 21.1.

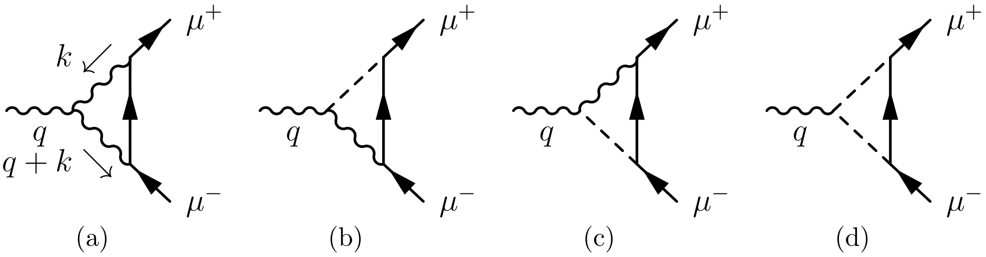


Figure 21.1: The weak-interaction contributions to muon's EM vertex. These four diagrams contain neutrino internal lines in the loops.

The Fig. 21.1(a) with W-neutrino loop reads

$$\begin{aligned}
\delta_\nu^{(a)}\Gamma^\mu(q) =& \frac{(\mathrm{i}g)^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\big[g^{\rho\lambda}(2k+q)^\mu + g^{\lambda\mu}(-2q-k)^\rho + g^{\rho\mu}(q-k)^\lambda\big] \\
&\times\frac{-\mathrm{i}g_{\rho\sigma}}{k^2-m_W^2}\frac{-\mathrm{i}g_{\lambda\kappa}}{(q+k)^2-m_W^2}\bar{u}(p')\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\gamma^\kappa\Big(\frac{1-\gamma^5}{2}\Big)u(p) \\
=& \frac{\mathrm{i}g^2}{2}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\bar{u}(p')\Big[(2k+q)^\mu\gamma^\sigma(\not{p}'+\not{k})\gamma_\sigma
\end{aligned}$$

$$+ (-\not{k} - 2\not{q})(\not{p}' + \not{k})\gamma^\mu + \gamma^\mu(\not{p}' + \not{k})(\not{q} - \not{k})\Big]\Big(\frac{1-\gamma^5}{2}\Big)u(p), \tag{21.1}$$

where

$$k' = k + xq + yp',$$
$$\Delta = (1-y)m_W^2 - x(1-x)q^2 - y(1-y)p'^2 + 2xyq\cdot p'.$$

To extract the form factor $F_2(q^2)$, recall that the total diagram can be written as a linear combination of $(p'+p)^\mu$, q^μ, γ^μ, and parity-violating terms containing γ^5. Only the $(p'+p)^\mu$ terms contribute to $F_2(q^2)$ through the Gordon identity. With this in mind, now we try to simplify the expression in the square bracket in (21.1), during which we will drops terms proportional to q^μ or γ^μ freely, and totally ignore the γ^5 terms.

$$\begin{aligned}
&\Big[(2k+q)^\mu\gamma^\sigma(\not{p}'+\not{k})\gamma_\sigma\Big] + \Big[(-\not{k}-2\not{q})(\not{p}'+\not{k})\gamma^\mu\Big] + \Big[\gamma^\mu(\not{p}'+\not{k})(\not{q}-\not{k})\Big]\\
=&\Big[-2\big(2k'+(1-2x)q-2yp'\big)^\mu\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\Big]\\
&+\Big[\big(-\not{k}'-(2-x)\not{q}+y\not{p}'\big)\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\gamma^\mu\Big]\\
&+\Big[\gamma^\mu\big(\not{k}'-x\not{q}+(1-y)\not{p}'\big)\big(-\not{k}'+(1+x)\not{q}+y\not{p}'\big)\Big]\\
\Rightarrow&\Big[4y(1-y)mp'^\mu\Big] + \Big[2(x+2y-2)mp^\mu\Big] + \Big[2(-1-x+y)mp'^\mu\Big]\\
\Rightarrow&-(1-y)(3-2y)m(p'+p)^\mu\\
\Rightarrow&\,2(1-y)(3-2y)m^2\cdot\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}.
\end{aligned}$$

The steps of this calculation is basically in parallel with the one of Problem 7.2. Here we have written the mass of muon as m instead of m_μ to avoid confusions. Thus the contribution to the muon's AMM from Fig. 21.1(a) is

$$\begin{aligned}
&\frac{\mathrm{i}g^2}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\cdot\frac{1}{2}\cdot 2(1-y)(3-2y)m^2\\
\simeq&\,\frac{7}{3}\cdot\frac{g^2m^2}{64\pi^2m_W^2} = \frac{7}{3}\cdot\frac{G_Fm^2}{8\pi^2\sqrt{2}},
\end{aligned} \tag{21.2}$$

where we have used the approximation $m_W \gg m$, and set $q^2 = 0$ in the second line. The Fermi constant $G_F/\sqrt{2} = g^2/8m_W^2$.

Fig. 21.1(b) and 21.1(c) read

$$\begin{aligned}
\delta_\delta^{(b)}\Gamma^\mu(q) =&\,\frac{\mathrm{i}g}{\sqrt{2}}\cdot\frac{-m_W}{2}\cdot\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,g^{\mu\rho}\frac{\mathrm{i}}{k^2-m_W^2}\frac{-\mathrm{i}g_{\rho\sigma}}{(q+k)^2-m_W^2}\\
&\times\bar{u}(p')\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)u(p).
\end{aligned} \tag{21.3}$$

$$\delta_\delta^{(c)}\Gamma^\mu(q) = \frac{\mathrm{i}g}{\sqrt{2}}\cdot\frac{-m_W}{2}\cdot\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,g^{\mu\rho}\frac{-\mathrm{i}g_{\rho\sigma}}{k^2-m_W^2}\frac{\mathrm{i}}{(q+k)^2-m_W^2}$$

$$\times\,\bar{u}(p')\gamma^\sigma\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\Big(\frac{1+\gamma^5}{2}\Big)u(p). \tag{21.4}$$

Through the calculation similar to that of Fig. 21.1(a), it is easy to show that these two diagrams contribute the same to the AMM, which reads

$$\frac{1}{2}\cdot\frac{G_F m^2}{8\pi^2\sqrt{2}}. \tag{21.5}$$

Finally, Fig. 21.1(d) reads

$$\begin{aligned}\delta_\delta^{(d)}\Gamma^\mu(q)=&\Big(\frac{-\mathrm{i}\sqrt{2}gm}{m_W}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}(2k-q)^\mu\frac{\mathrm{i}}{k^2-m_W^2}\frac{\mathrm{i}}{(q+k)^2-m_W^2}\\&\times\bar{u}(p')\Big(\frac{1-\gamma^5}{2}\Big)\frac{\mathrm{i}}{\not{p}'+\not{k}}\Big(\frac{1+\gamma^5}{2}\Big)u(p).\end{aligned} \tag{21.6}$$

But it is not difficult to see that the contribution to the muon's AMM from this diagram is proportional to $(m/m_W)^4$, which can be omitted in the limit $m_W \gg m$, compared with the other three diagrams. Therefore we conclude that the AMM of the muon contributed by W-neutrino and corresponding Goldstone boson's 1-loop diagrams is

$$a_\mu(\nu)=\Big[\frac{7}{3}+\frac{1}{2}+\frac{1}{2}+\mathcal{O}\Big(\frac{m^2}{m_W^2}\Big)\Big]\cdot\frac{G_Fm^2}{8\pi^2\sqrt{2}}\simeq\frac{10}{3}\cdot\frac{G_Fm^2}{8\pi^2\sqrt{2}}. \tag{21.7}$$

(c) Now we come to the second set of diagrams as shown in Fig. 21.2.

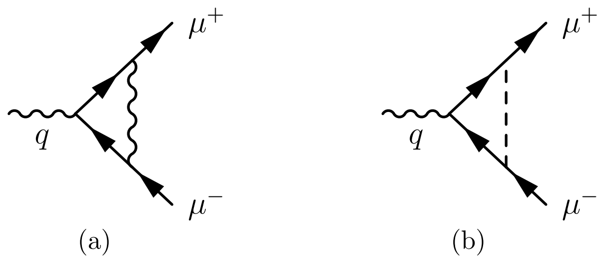


Figure 21.2: The weak-interaction contributions to muon's EM vertex. These two diagrams contains no neutrino internal lines.

Firstly the Fig. 21.2(a) reads

$$\begin{aligned}\delta_Z^{(a)}\Gamma^\mu(q)=&\Big(\frac{\mathrm{i}g}{4c_w}\Big)^2\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{-\mathrm{i}g_{\rho\sigma}}{(p'+k)^2-m_Z^2}\\&\times\bar{u}(p')\gamma^\rho\big(4s_w^2-1-\gamma^5\big)\frac{\mathrm{i}}{-\not{k}-m}\gamma^\mu\frac{\mathrm{i}}{-\not{q}-\not{k}-m}\gamma^\sigma\big(4s_w^2-1-\gamma^5\big)u(p)\\\Rightarrow&\frac{-\mathrm{i}g^2}{16c_w^2}\int\frac{\mathrm{d}^dk'}{(2\pi)^d}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2}{(k'^2-\Delta)^3}\bar{u}(p')\gamma^\rho\Big[(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+m)\\&+(4s_w^2-1)^2(\not{k}-m)\gamma^\mu(\not{k}+\not{q}-m)\Big]\gamma_\rho u(p),\end{aligned} \tag{21.8}$$

where we have omitted terms proportional to γ^5, as indicated by "$\Rightarrow$" sign, and

$$\begin{aligned} k' &= k + xq + yp', \\ \Delta &= (1-y)m^2 + ym_Z^2 - x(1-x)q^2 - y(1-y)p'^2 + 2xyq\cdot p'. \end{aligned}$$

We will again focus only on the terms proportional to $(p'+p)^\mu$. Then the spinor part can be reduced to

$$\begin{aligned} &\bar{u}(p')\gamma^\rho\Big[(\not{k}+m)\gamma^\mu(\not{k}+\not{q}+m) + (3s_w^2 - c_w^2)^2(\not{k}-m)\gamma^\mu(\not{k}+\not{q}-m)\Big]\gamma_\rho u(p) \\ \Rightarrow\ &\Big[2y(3+y) - (4s_w^2-1)^2\cdot 2y(1-y)\Big]2m^2\cdot\bar{u}(p')\frac{\mathrm{i}\sigma^{\mu\nu}q_\nu}{2m}u(p). \end{aligned}$$

Thus the AMM contribute by this diagram is

$$\begin{aligned} &-\frac{\mathrm{i}g^2}{16c_w^2}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{2\cdot 2m^2[2y(3+y)-(4s_w^2-1)^2\cdot 2y(1-y)]}{(k'^2-\Delta)^3} \\ =\ &\frac{G_F m^2}{8\pi^2\sqrt{2}}\cdot\frac{1}{3}\Big[(4s_w^2-1)^2 - 5\Big]. \end{aligned} \tag{21.9}$$

On the other hand, the Fig. 21.2(b) only contributes terms of order m^4/m_W^4 that can be omitted, as can be seen from the coupling between the Goldstone boson and the muon. Thus we conclude that the total contribution to $a_\mu(Z)$ from the two diagrams in Fig. 21.2 at the leading order is given by (21.9).

21.2 Complete analysis of $e^+e^- \to W^+W^-$

In this problem we calculate the amplitude for the process $e^+e^- \to W^+W^-$ at tree level in standard electroweak theory. There are 3 diagrams contributing in total, as shown in Figure 21.3.

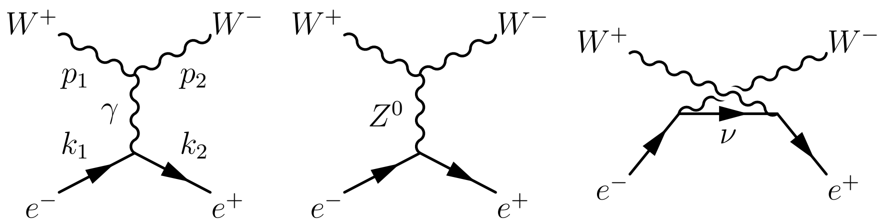


Figure 21.3: The process $e^-e^+ \to W^+W^-$ at tree level. All initial momenta go inward and all final momenta go outward.

We will evaluate these diagrams for definite helicities for initial electrons as well as definite polarizations for final W bosons. The initial and final momenta can be parameterized as

$$\begin{aligned} k_1^\mu &= (E,0,0,E), & p_1^\mu &= (E, p\sin\theta, 0, p\cos\theta), \\ k_2^\mu &= (E,0,0,-E), & p_2^\mu &= (E, -p\sin\theta, 0, -p\cos\theta), \end{aligned} \tag{21.10}$$

with $E^2 = p^2 + m_W^2$, and electron mass ignored. For initial electron and positron, the spinors with definite helicities can be chosen to be

$$\begin{aligned} u_L(k_1) &= \sqrt{2E}(0,1,0,0)^T, & v_L(k_2) &= \sqrt{2E}(1,0,0,0)^T, \\ u_R(k_1) &= \sqrt{2E}(0,0,1,0)^T, & v_R(k_2) &= \sqrt{2E}(0,0,0,1)^T. \end{aligned} \tag{21.11}$$

For final W bosons, the polarization vectors are

$$\begin{aligned} \epsilon^*_{-\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0,-\cos\theta,-\mathrm{i},\sin\theta), & \epsilon^*_{-\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0,\cos\theta,-\mathrm{i},-\sin\theta), \\ \epsilon^*_{+\mu}(p_1) &= \tfrac{1}{\sqrt{2}}(0,-\cos\theta,\mathrm{i},\sin\theta), & \epsilon^*_{+\mu}(p_2) &= \tfrac{1}{\sqrt{2}}(0,\cos\theta,\mathrm{i},-\sin\theta), \\ \epsilon^*_{L\mu}(p_1) &= \tfrac{1}{m_W}(p,-E\sin\theta,0,-E\cos\theta), & \epsilon^*_{L\mu}(p_2) &= \tfrac{1}{m_W}(p,E\sin\theta,0,E\cos\theta). \end{aligned} \tag{21.12}$$

It is easy to see that for initial electron-positron pair, only two helicity states $e^-_L e^+_R$ and $e^-_R e^+_L$ contribute nonzero amplitudes. This is because the first two diagrams with s-channel gauge bosons vanish for the other two possibilities $e^-_L e^+_L$ and $e^-_R e^+_R$ due to angular momentum conservation, while the third diagram vanishes since the weak coupling vanishes for right-handed electron and left-handed positron. With this known, we can write down the amplitudes for $e^-_L e^+_R$ and $e^-_R e^+_L$ initial states, as follows. Generally the amplitude reads

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e^-_L e^+_R \to W^+W^-) \\ &= \bigg\{\bigg[(-\mathrm{i}e)\frac{-\mathrm{i}}{(k_1+k_2)^2}(\mathrm{i}e) + \frac{\mathrm{i}e(-\frac{1}{2}+s_w^2)}{c_w s_w}\frac{-\mathrm{i}}{(k_1+k_2)^2-m_Z^2}(\mathrm{i}gc_w)\bigg] \\ &\quad \times \bar{v}_L(k_2)\gamma_\lambda u_L(k_1)\big[\eta^{\mu\nu}(p_2-p_1)^\lambda + \eta^{\nu\lambda}(-p_1-2p_2)^\mu + \eta^{\lambda\mu}(2p_1+p_2)^\nu\big] \\ &\quad + \Big(\frac{\mathrm{i}g}{\sqrt{2}}\Big)^2 \bar{v}_L(k_2)\gamma^\mu \frac{\mathrm{i}}{\not{k}_1 - \not{p}_2}\gamma^\nu u_L(k_1)\bigg\}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) \\ &= \mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)} - \frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\bar{v}_L(k_2)\Big(\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not{p}_2-\not{p}_1) \\ &\qquad - (p_1+2p_2)\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2) + (2p_1+p_2)\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\Big)u_L(k_1) \\ &\quad - \frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot\bar{v}_L(k_2)\not{\epsilon}^*(p_1)(\not{k}_1-\not{p}_2)\not{\epsilon}^*(p_2)u_L(k_1), \end{aligned} \tag{21.13}$$

and,

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e^-_R e^+_L \to W^+W^-) \\ &= \bigg[(-\mathrm{i}e)\frac{-\mathrm{i}}{(k_1+k_2)^2}(\mathrm{i}e) + \frac{\mathrm{i}es_w^2}{c_w s_w}\frac{-\mathrm{i}}{(k_1+k_2)^2-m_Z^2}(\mathrm{i}gc_w)\bigg]\bar{v}_R(k_2)\gamma_\lambda u_R(k_1) \\ &\quad \times \big[\eta^{\mu\nu}(p_2-p_1)^\lambda + \eta^{\nu\lambda}(-p_1-2p_2)^\mu + \eta^{\lambda\mu}(2p_1+p_2)^\nu\big]\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) \\ &= \mathrm{i}e^2\frac{m_Z^2}{s(s-m_Z^2)}\bar{v}_R(k_2)\Big(\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not{p}_2-\not{p}_1) - (p_1+2p_2)\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2) \\ &\qquad + (2p_1+p_2)\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\Big)u_R(k_1), \end{aligned} \tag{21.14}$$

In what follows we need the inner products among some of these vectors, as listed below.

$$p_1\cdot p_2 = E^2+p^2 \qquad p_1\cdot\epsilon_0^*(p_2) = p_2\cdot\epsilon_0^*(p_1) = \frac{2Ep}{m_W},$$
$$\epsilon_+^*(p_1)\cdot\epsilon_+^*(p_2) = \epsilon_-^*(p_1)\cdot\epsilon_-^*(p_2) = 1, \qquad \epsilon_0^*(p_1)\cdot\epsilon_0^*(p_2) = \frac{E^2+p^2}{m_W^2}. \tag{21.15}$$

We also need

$$\bar{v}_L(k_2)\not{p}_1 u_L(k_1) = -\bar{v}_L(k_2)\not{p}_2 u_L(k_1) = 2Ep\sin\theta, \tag{21.16}$$
$$\bar{v}_L(k_2)\not{\epsilon}_\pm^*(p_1)u_L(k_1) = 2E\Big(\frac{\mp 1+\cos\theta}{\sqrt{2}}\Big), \tag{21.17}$$
$$\bar{v}_L(k_2)\not{\epsilon}_\pm^*(p_2)u_L(k_1) = 2E\Big(\frac{\mp 1-\cos\theta}{\sqrt{2}}\Big), \tag{21.18}$$
$$\bar{v}_L(k_2)\not{\epsilon}_0^*(p_1)u_L(k_1) = -\bar{v}_L(k_2)\not{\epsilon}_0^*(p_2)u(k_1) = \frac{2E^2\sin\theta}{m_W}, \tag{21.19}$$
$$\bar{v}_R(k_2)\not{p}_1 u_R(k_1) = -\bar{v}_R(k_2)\not{p}_2 u_R(k_1) = -2Ep\sin\theta, \tag{21.20}$$
$$\bar{v}_L(k_2)\not{\epsilon}_\pm^*(p_1)u_L(k_1) = -2E\Big(\frac{\mp 1+\cos\theta}{\sqrt{2}}\Big), \tag{21.21}$$
$$\bar{v}_L(k_2)\not{\epsilon}_\pm^*(p_2)u_L(k_1) = -2E\Big(\frac{\mp 1-\cos\theta}{\sqrt{2}}\Big), \tag{21.22}$$
$$\bar{v}_L(k_2)\not{\epsilon}_0^*(p_1)u_L(k_1) = -\bar{v}_L(k_2)\not{\epsilon}_0^*(p_2)u(k_1) = -\frac{2E^2\sin\theta}{m_W}. \tag{21.23}$$

We first consider $e_L^-e_R^+\to W^+W^-$. In this case we take $u(k_1)=u_L(k_1)=\sqrt{2E}(0,1,0,0)^T$ and $\bar{v}(k_2)=\bar{v}_L(k_2)=\sqrt{2E}(0,0,1,0)$. Then each of the final W particle can have polarization $(+,-,0)$, which gives 9 possible combinations for (W^+,W^-). Now we evaluate the corresponding amplitudes in turn.

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(0)}^+W_{(0)}^-)\\
=\,&\mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)}-\frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\Big(-\frac{4Ep(E^2+p^2)}{m_W^2}+\frac{16E^3p}{m_W^2}\Big)\sin\theta\\
&+\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot\frac{2E(-3E^2p+p^3-2E^3\cos\theta)\sin\theta}{m_W^2}\\
=&-\mathrm{i}e^2\cdot\frac{s}{4m_W^2}\bigg\{\frac{m_Z^2}{s-m_Z^2}\cdot\beta(3-\beta^2)\\
&-\frac{1}{2s_w^2}\bigg[\Big(\frac{2}{1+\beta^2+2\beta\cos\theta}-\frac{s}{s-m_Z^2}\Big)\beta(3-\beta^2)+\frac{4\cos\theta}{1+\beta^2+2\beta\cos\theta}\bigg]\bigg\}\sin\theta
\end{aligned}\tag{21.24}$$

$$\begin{aligned}
&\mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(0)}^+W_{(\pm)}^-) = \mathrm{i}\mathcal{M}(e_L^-e_R^+\to W_{(\mp)}^+W_{(0)}^-)\\
=\,&\mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)}-\frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\Big(\frac{8E^2p}{m_W}\frac{\mp 1+\cos\theta}{\sqrt{2}}\Big)\\
&-\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot\frac{2E}{m_W}(E^2(2\cos\theta\mp 1)+2Ep\pm p^2)\frac{\pm 1+\cos\theta}{\sqrt{2}}
\end{aligned}$$

$$= \mathrm{i}e^2\bigg[\frac{m_Z^2}{s-m_Z^2}\beta - \frac{1}{2s_W^2}\bigg(\frac{s}{s-m_Z^2}\beta + \frac{\pm 1 - 2\cos\theta - 2\beta \mp \beta^2}{1+\beta^2+2\beta\cos\theta}\bigg)\bigg]\frac{\sqrt{s}}{m_W}\frac{\pm 1 + \cos\theta}{\sqrt{2}} \tag{21.25}$$

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(\pm)}^+ W_{(\pm)}^-) \\ &= \mathrm{i}e^2\bigg[\frac{m_Z^2}{s(s-m_Z^2)} - \frac{1}{2s_w^2}\frac{1}{s-M_Z^2}\bigg]\Big(-4Ep\sin\theta\Big) + \frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot 2E(p+E\cos\theta)\sin\theta \\ &= \mathrm{i}e^2\bigg[-\frac{m_Z^2}{(s-m_Z^2)}\beta + \frac{1}{2s_w^2}\bigg(\frac{s}{s-M_Z^2}\beta - \frac{2(\beta+\cos\theta)}{(1+\beta^2+2\beta\cos\theta)}\bigg)\bigg]\sin\theta \end{aligned} \tag{21.26}$$

$$\begin{aligned} &\mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(\pm)}^+ W_{(\mp)}^-) \\ &= -\frac{\mathrm{i}e^2}{2s_w^2}\frac{1}{u}\cdot 2E^2(\mp 1 + \cos\theta)\sin\theta = \frac{\mathrm{i}e^2}{2s_w^2}\frac{2(\pm 1 - \cos\theta)\sin\theta}{(1+\beta^2+2\beta\cos\theta)}. \end{aligned} \tag{21.27}$$

Though not manifest, these expressions have correct high energy behavior. To see this, we note that $\beta \simeq 1 - 2m_W^2/s$ when $s \gg m_W^2$. Then, for instance, the amplitude for two longitudinal W final state becomes

$$\begin{aligned} \mathrm{i}\mathcal{M}(e_L^- e_R^+ \to W_{(0)}^+ W_{(0)}^-) &= -\mathrm{i}e^2 \cdot \frac{s}{4m_W^2}\bigg\{\frac{m_Z^2}{s-m_Z^2}\cdot\beta(3-\beta^2) \\ &- \frac{1}{2s_w^2}\bigg[\Big(\frac{2}{1+\beta^2+2\beta\cos\theta} - \frac{s}{s-m_Z^2}\Big)\beta(3-\beta^2) + \frac{4\cos\theta}{1+\beta^2+2\beta\cos\theta}\bigg]\bigg\}\sin\theta \\ &= -\frac{\mathrm{i}e^2}{2s_W^2}\frac{(1+2\cos\theta)\sin\theta}{1+\cos\theta} + \mathcal{O}(1/s). \end{aligned} \tag{21.28}$$

Then we can plot the azimuthal distribution of the corresponding differential cross section at $s = (1000\text{GeV})^2$, as shown in Figure 21.4.

Next we consider the other case with $e_R^- e_L^+$ initial state. Now there is no contribution from u-channel neutrino exchange. The amplitudes for various polarizations of final W pairs can be worked out to be

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(0)}^+ W_{(0)}^-) = \mathrm{i}e^2\frac{s}{s-m_Z^2}\frac{m_Z^2}{4m_W^2}\beta(\beta^2-3)\sin\theta, \tag{21.29}$$

$$\begin{aligned} \mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(0)}^+ W_{(\pm)}^-) &= \mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\mp)}^+ W_{(0)}^-) \\ &= \mathrm{i}e^2\frac{m_Z^2}{s-m_Z^2}\frac{\sqrt{s}}{m_W}\beta\frac{\pm 1 - \cos\theta}{\sqrt{2}} \end{aligned} \tag{21.30}$$

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\pm)}^+ W_{(\pm)}^-) = \mathrm{i}e^2\frac{m_Z^2}{s-m_Z^2}\beta\sin\theta \tag{21.31}$$

$$\mathrm{i}\mathcal{M}(e_R^- e_L^+ \to W_{(\pm)}^+ W_{(\mp)}^-) = 0. \tag{21.32}$$

21.3 Cross section for $d\bar{u} \to W^-\gamma$

In this problem we compute the tree amplitude of $d\bar{u} \to W^-\gamma$ at high energies so that the quark masses can be ignored. In this case the left-handed and right-handed spinors

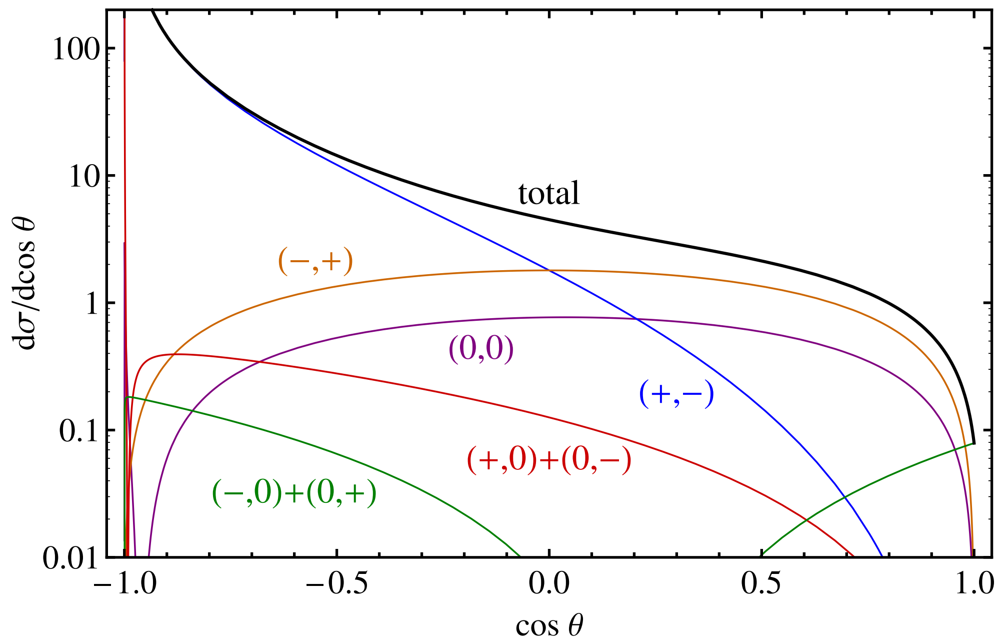


Figure 21.4: The differential cross section of $e^-e^+ \to W^+W^-$ with definite helicity as a function of azimuthal angle at $s = (1000\text{GeV})^2$.

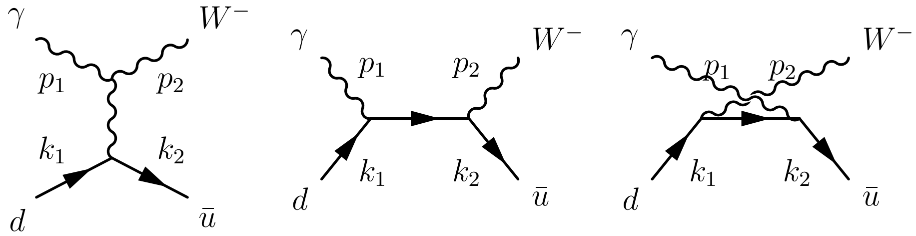


Figure 21.5: The process $d\bar{u} \to W^-\gamma$ at tree level. All initial momenta go inward and all final momenta go outward.

decouple and only the amplitudes with $d_L\bar{u}_R$ initial state do not vanish. To calculate it, we firstly work out the kinematics as follows.

$$\begin{aligned} k_1 &= (E, 0, 0, E), & p_1 &= (p, p\sin\theta, 0, p\cos\theta), \\ k_2 &= (E, 0, 0, -E), & p_2 &= (E_W, -p\sin\theta, 0, -p\cos\theta), \end{aligned} \tag{21.33}$$

where $p = E - m_W^2/4E$ and $E_W = E + m_W^2/4E$. The initial spinors of definite helicities are given by

$$u_L(k_1) = \sqrt{2E}(0, 1, 0, 0)^T, \qquad v_L(k_2) = \sqrt{2E}(1, 0, 0, 0)^T, \tag{21.34}$$

while the polarization vectors for final photon and W^- read

$$\begin{aligned} &\epsilon^*_{\pm\mu}(p_1) = \tfrac{1}{\sqrt{2}}(0, -\cos\theta, \pm\mathrm{i}, \sin\theta), \qquad \epsilon^*_{\pm\mu}(p_2) = \tfrac{1}{\sqrt{2}}(0, \cos\theta, \pm\mathrm{i}, -\sin\theta), \\ &\epsilon^*_{L\mu}(p_2) = \tfrac{1}{m_W}(p, E_W\sin\theta, 0, E_W\cos\theta). \end{aligned} \tag{21.35}$$

Then the amplitude is given by

$$\mathrm{i}\mathcal{M}(d_L\bar{u}_R \to \gamma W^-) = \frac{-\mathrm{i}e^2}{\sqrt{2}s_w}\frac{N_s}{s - m_W^2} - \frac{\mathrm{i}e^2}{3\sqrt{2}s_w}\left(\frac{-N_t}{t} + \frac{2N_u}{u}\right), \tag{21.36}$$

where

$$\begin{aligned}N_s &= \bar{v}_L(k_2)\big[\epsilon^*(p_1)\cdot\epsilon^*(p_2)(\not{p}_1-\not{p}_2)+(p_1+2p_2)\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2)\\ &\qquad -(2p_1+p_2)\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\big]u_L(k_1), \qquad (21.37)\\ N_t &= \bar{v}_L(k_2)\not{\epsilon}^*(p_2)(\not{k}_1-\not{p}_1)\not{\epsilon}^*(p_1)u_L(k_1), \qquad (21.38)\\ N_u &= \bar{v}_L(k_2)\not{\epsilon}^*(p_1)(\not{k}_1-\not{p}_2)\not{\epsilon}^*(p_2)u_L(k_1). \qquad (21.39)\end{aligned}$$

Now, using the physical conditions $\epsilon^*(p_i)\cdot p_i=0$, $\not{k}_1u_L(k_1)=0$ and $\bar{v}_L(k_2)\not{k}_2=0$, we can show that $N_s=N_t-N_u$. In fact,

$$\begin{aligned}N_s &= \bar{v}_L(k_2)\big[2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\not{p}_1+2p_2\cdot\epsilon^*(p_1)\not{\epsilon}^*(p_2)-2p_1\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)\big]u_L(k_1),\\ N_t &= \bar{v}_L(k_2)\big[2k_1\cdot\epsilon_1\not{\epsilon}_2+2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\not{p}_1-\not{\epsilon}^*(p_1)\not{\epsilon}^*(p_2)\not{p}_1\big]u_L(k_1),\\ N_u &= \bar{v}_L(k_2)\big[-2k_2\cdot\epsilon_1\not{\epsilon}_2+2p_1\cdot\epsilon^*(p_2)\not{\epsilon}^*(p_1)-\not{\epsilon}^*(p_1)\not{\epsilon}^*(p_2)\not{p}_1\big]u_L(k_1).\end{aligned}$$

Then $N_s=N_t-N_u$ is manifest. Note further that $s-m_W^2=-(t+u)$, we have

$$\begin{aligned}\mathrm{i}\mathcal{M}(d_L\bar{u}_R\to\gamma W^-) &= \frac{\mathrm{i}e^2}{\sqrt{2}s_w}\bigg(\frac{N_t-N_u}{t+u}-\frac{N_t}{3t}+\frac{2N_u}{3u}\bigg)\\ &= \frac{\mathrm{i}e^2}{\sqrt{2}s_w}\frac{(2t-u)}{3(t+u)}\Big(\frac{N_t}{t}+\frac{N_u}{u}\Big)\\ &= \frac{\mathrm{i}e^2}{6\sqrt{2}s_w}(1-3\cos\theta)\Big(\frac{N_t}{t}+\frac{N_u}{u}\Big). \qquad (21.40)\end{aligned}$$

One can see clearly from this expression that all helicity amplitudes vanish at $\cos\theta=1/3$. (Note that the definition of scattering angle θ is different from the one in Peskin & Schroeder, which, in our notation, is $\pi-\theta$.) Then, by including all helicity combinations (6 in total), we find the differential cross section, as a function of s and θ, to be

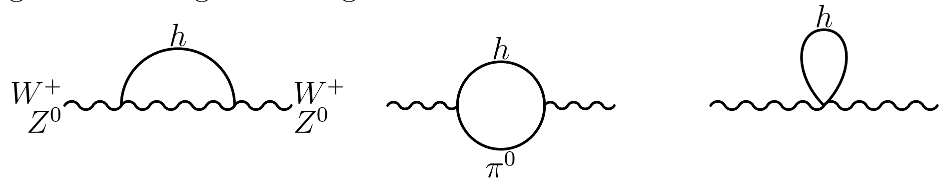
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}=\frac{\pi\alpha^2}{32s_w^2}\Big(\frac{1-\cos\theta}{\sin\theta}\Big)^2\frac{x^3+18x^2+9x+24-(x^3-14x^2+9x-8)\cos2\theta}{36(s-m_W^2)}, \qquad (21.41)$$

where $x\equiv m_W^2/s$.

21.4 Dependence of radiative corrections on the Higgs boson mass

(a) We first analyze the radiative corrections to μ decay process at 1-loop level with the Higgs boson in the loop. It is easy to see that if the internal Higgs boson line is attached to one of the external fermions, the resulted vertex will contribute a factor of m_f/v which can be ignored. Therefore only the vacuum polarization diagrams are relevant, and they should sum to a gauge invariant result.

(b) Now we compute the vacuum polarization amplitudes of $W^\pm$, Z^0 and photon with Higgs contribution. We will only consider the pieces proportional to $g^{\mu\nu}$, namely $\Pi_{WW}(q^2)$, $\Pi_{ZZ}(q^2)$, $\Pi_{\gamma\gamma}(q^2)$ and $\Pi_{Z\gamma}(q^2)$. It is easy to show that $\Pi_{\gamma\gamma}(q^2)$ and $\Pi_{ZZ}(q^2)$ receive no contribution from Higgs boson at 1-loop level, while $\Pi_{WW}(q^2)$ and $\Pi_{ZZ}(q^2)$ can be found by computing the following three diagrams:



Now we compute these three diagrams in turn for W^+. The first diagram reads

$$
\begin{aligned}
&(\mathrm{i}gm_W)^2 g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}\frac{-\mathrm{i}}{(q-k)^2-m_W^2}\\
=&-\frac{\mathrm{i}}{(4\pi)^2}g^2m_W^2g^{\mu\nu}\int_0^1\mathrm{d}x\,\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}(m_W^2,q^2)}\\
\Rightarrow&-\frac{\mathrm{i}}{(4\pi)^2}g^2m_W^2g^{\mu\nu}\Big[E+\int_0^1\mathrm{d}x\,\log\frac{M^2}{\Delta(m_W^2,q^2)}\Big],
\end{aligned}
\tag{21.42}
$$

where $\Delta(m_W^2,q^2)=xm_W^2+(1-x)m_h^2-x(1-x)q^2$, $E=2/\epsilon-\gamma+\log 4\pi-\log M^2$, and M^2 is the subtraction scale. The second one reads

$$
\begin{aligned}
&(\mathrm{i}g/2)^2\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}\frac{-\mathrm{i}}{(q-k)^2-m_W^2}(2k-p)^\mu(2k-p)^\nu\\
\Rightarrow&\,\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k'}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{(4/d)k'^2}{(k'^2-\Delta(m_W^2,q^2))^2}\\
\Rightarrow&\,\frac{\mathrm{i}}{(4\pi)^2}\frac{g^2}{4}g^{\mu\nu}\int_0^1\mathrm{d}x\,2\Delta(m_W^2,q^2)\Big[E+1+\log\frac{M^2}{\Delta(m_W^2,q^2)}\Big],
\end{aligned}
\tag{21.43}
$$

in which we have ignored terms proportional to $q^\mu q^\nu$. Then, the last diagram reads

$$
\begin{aligned}
&\frac{1}{2}(\mathrm{i}g^2/2)g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-m_h^2}=-\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-m_h^2}\frac{(q-k)^2-m_W^2}{(q-k)^2-m_W^2}\\
=&-\frac{g^2}{4}g^{\mu\nu}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1\mathrm{d}x\frac{k'^2+(1-x)^2q^2-m_W^2}{(k'^2-\Delta(m_W^2,q^2))^2}\\
\Rightarrow&-\frac{\mathrm{i}}{(4\pi)^2}\frac{g^2}{4}g^{\mu\nu}\int_0^1\mathrm{d}x\,\bigg[\Big(2\Delta(m_W^2,q^2)-m_W^2+(1-x)^2q^2\Big)E\\
&+\Big(2\Delta(m_W^2,q^2)-m_W^2+(1-x)^2q^2\Big)\log\frac{M^2}{\Delta(m_W^2,q^2)}+\Delta(m_W^2,q^2)\bigg].
\end{aligned}
\tag{21.44}
$$

Thus we have, when the three diagrams above are taken into account only,

$$
\begin{aligned}
\Pi_{WW}(q^2)=&\frac{g^2}{4(4\pi)^2}\bigg[-\Big(3m_W^2+\frac{1}{3}q^2\Big)E\\
&+\int_0^1\mathrm{d}x\,\bigg(\Delta(m_W^2,q^2)-\big[3m_W^2+(1-x)^2q^2\big]\log\frac{M^2}{\Delta(m_W^2,q^2)}\bigg)\bigg].
\end{aligned}
\tag{21.45}
$$

Now we extract Higgs mass contribution from this expression in the large Higgs limit, and also fix the subtraction point at $M^2 = m_W^2$. In this limit we may take $\Delta(m_W^2, q^2) \simeq x m_h^2$, and $\log(M^2/\Delta) \simeq -\log(m_h^2/m_W^2)$. We also throw divergent terms with E, which should be canceled out in the final expression of zeroth order natural relation after including completely loop diagrams with W, Z, and would-be Goldstone boson internal lines. Then we have

$$\Pi_{WW}(q^2) = \frac{g^2}{4(4\pi)^2}\left[\frac{1}{2}m_h^2 + \Big(3m_W^2 + \frac{1}{3}q^2\Big)\log\frac{m_h^2}{m_W^2}\right]. \tag{21.46}$$

Similarly, we have, for $\Pi_{ZZ}(q^2)$,

$$\Pi_{ZZ}(q^2) = \frac{g^2}{4(4\pi)^2\cos^2\theta_w}\left[\frac{1}{2}m_h^2 + \Big(3m_Z^2 + \frac{1}{3}q^2\Big)\log\frac{m_h^2}{m_Z^2}\right]. \tag{21.47}$$

(c) Now, we derive the zeroth order natural relation given in (21.134) of Peskin & Schroeder, in the large Higgs mass limit. Note that $\Pi_{\gamma\gamma} = \Pi_{Z\gamma} = 0$. Thus,

$$\begin{aligned} s_*^2 - \sin^2\theta_0^2 &= \frac{\sin^2\theta_w\cos^2\theta_w}{\cos^2\theta_w - \sin^2\theta_w}\left(\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2}\right) \\ &= \frac{\alpha}{48\pi}\frac{1+9\sin^2\theta_w}{\cos^2\theta_w - \sin^2\theta_w}\log\frac{m_h^2}{m_W^2}, \end{aligned} \tag{21.48}$$

$$s_W^2 - s_*^2 = -\frac{\Pi_{WW}(m_W^2)}{m_Z^2} + \frac{m_W^2}{m_Z^2}\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} = \frac{5\alpha}{24\pi}\log\frac{m_h^2}{m_W^2}. \tag{21.49}$$

Final Project III
Decays of the Higgs Boson

In this final project, we calculate partial widths of various decay channels of the standard model Higgs boson. Although a standard-model-Higgs-like boson has been found at the LHC with mass around 125GeV, it is still instructive to treat the mass of the Higgs boson as a free parameter in the following calculation.

The main decay modes of Higgs boson include $h^0 \to f\bar{f}$ with f the standard model fermions, $h^0 \to W^+W^-$, $h^0 \to Z^0Z^0$, $h^0 \to gg$ and $h^0 \to \gamma\gamma$. The former three processes appear at the tree level, while the leading order contributions to the latter two processes are at one-loop level. We will work out the decay widths of these processes in the following.

In this problem we only consider the two-body final states. The calculation of decay width needs the integral over the phase space of the two-body final states. By momentum conservation and rotational symmetry, we can always parameterize the momenta of two final particles in CM frame to be $p_1 = (E, 0, 0, p)$ and $p_2 = (E, 0, 0, -p)$, where $E = \frac{1}{2}m_h$ by energy conservation. Then the amplitude $\mathcal{M}$ will have no angular dependence. Then the phase space integral reads

$$\int \mathrm{d}\Pi_2 \, |\mathcal{M}|^2 = \frac{1}{4\pi}\frac{p}{m_h}|\mathcal{M}|^2. \tag{21.50}$$

Then the decay width is given by

$$\Gamma = \frac{1}{2m_h}\int \mathrm{d}\Pi_2 \, |\mathcal{M}|^2 = \frac{1}{8\pi}\frac{p}{m_h^2}|\mathcal{M}|^2. \tag{21.51}$$

In part (d) of this problem, we will also be dealing with the production of the Higgs boson from two-gluon initial state, thus we also write down the formula here for the cross section of the one-body final state from two identical initial particle. This time, the two ingoing particles have momenta $k_1 = (E, 0, 0, k)$ and $k_2 = (E, 0, 0, -k)$, with $E^2 = k^2 + m_i^2$ and $2E = m_f$ where m_i and m_f are masses of initial particles and final particle, respectively. The final particle has momentum $p = (m_f, 0, 0, 0)$. Then, the cross section is given by

$$\begin{aligned}
\sigma &= \frac{1}{2\beta s}\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{2E_p}|\mathcal{M}|^2(2\pi)^4\delta^{(4)}(p - k_1 - k_2) \\
&= \frac{1}{4m_f\beta s}|\mathcal{M}|^2(2\pi)\delta(2k - m_f) = \frac{\pi}{\beta m_f^2}|\mathcal{M}|^2\delta(s - m_f^2),
\end{aligned} \tag{21.52}$$

where $\beta = \sqrt{1-(4m_i/m_f)^2}$ is the magnitude of the velocity of the initial particle in the center-of-mass frame.

(a) The easiest calculation of above processes is $h^0 \to f\bar{f}$, where f represents all quarks and charged leptons. The tree level contribution to this process involves a single Yukawa vertex only. The corresponding amplitude is given by

$$\mathrm{i}\mathcal{M}(h^0 \to f\bar{f}) = -\frac{\mathrm{i}m_f}{v}\bar{u}^*(p_1)v(p_2). \tag{21.53}$$

Then it is straightforward to get the squared amplitude with final spins summed to be

$$\sum |\mathcal{M}(h^0 \to f\bar{f})|^2 = \frac{m_f^2}{v^2}\,\mathrm{tr}\left[(\not{p}_1 + m_f)(\not{p}_2 - m_f)\right] = \frac{2m_f^2}{v^2}(m_h^2 - 4m_f^2). \tag{21.54}$$

In CM frame, the final states momenta can be taken to be $p_1 = (E,0,0,p)$ and $p_2 = (E,0,0,-p)$, with $E = \frac{1}{2}m_h$ and $p^2 = E^2 - m_f^2$. Then the decay width is given by

$$\Gamma(h^0 \to f\bar{f}) = \frac{1}{8\pi}\frac{p}{m_h^2}|\mathcal{M}|^2 = \frac{m_h m_f^2}{8v^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}. \tag{21.55}$$

This expression can be expressed in terms of the fine structure constant α, the mass of W boson m_w and Weinberg angle $\sin\theta_w$, as

$$\Gamma(h^0 \to f\bar{f}) = \frac{\alpha m_h}{8\sin^2\theta_w}\frac{m_f^2}{m_W^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}. \tag{21.56}$$

(b) Next we consider the decay of h^0 to massive vector bosons W^+W^- and Z^0Z^0. The amplitude for the process $h^0 \to W^+W^-$ is given by

$$\mathrm{i}\mathcal{M}(h^0 \to W^+W^-) = \frac{\mathrm{i}g^{\mu\nu}g^2 v}{2}\epsilon_\mu^*(p_1)\epsilon_\nu^*(p_2). \tag{21.57}$$

Then the squared amplitude with final polarizations summed reads

$$\begin{aligned}\sum |\mathcal{M}|^2 &= \frac{g^4v^2}{4}\left(g_{\mu\nu} - \frac{p_{1\mu}p_{1\nu}}{m_W^2}\right)\left(g^{\mu\nu} - \frac{p_2^\mu p_2^\nu}{m_W^2}\right)\\ &= \frac{\pi\alpha}{\sin^2\theta_w}\frac{m_h^4}{m_W^2}\left(1 - \frac{4m_W^2}{m_h^2} + \frac{12m_W^4}{m_h^4}\right).\end{aligned} \tag{21.58}$$

Therefore the decay width is

$$\Gamma(h^0 \to W^+W^-) = \frac{1}{8\pi}\frac{p_1}{m_h^2}|\mathcal{M}|^2 = \frac{\alpha m_h^3}{16\pi m_W^2\sin^2\theta_w}(1 - 4\tau_W^{-1} + 12\tau_W^{-2})(1 - 4\tau_W^{-1})^{1/2}, \tag{21.59}$$

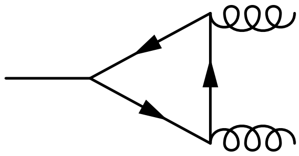
where we have defined $\tau_W \equiv (m_h/m_W)^2$ for brevity. For $h^0 \to Z^0Z^0$ process, it can be easily checked that nothing gets changed in the calculation except that all m_W should be replaced with m_Z, while an additional factor $1/2$ is needed to account for the identical particles in final state. Therefore we have

$$\Gamma(h^0 \to Z^0Z^0) = \frac{\alpha m_h^3}{32\pi m_Z^2\sin^2\theta_w}(1 - 4\tau_Z^{-1} + 12\tau_Z^{-2})(1 - 4\tau_Z^{-1})^{1/2}, \tag{21.60}$$

where $\tau_Z \equiv (m_h/m_Z)^2$.

The calculation above considered "on-shell" decay only, while for realistic 125GeV boson, the off-shell decay turns out to be very important. That is, although $h^0 \to W^+W^-$ and $h^0 \to Z^0 Z^0$ are kinetically forbidden when $m_h = 125\text{GeV}$ according to above results, the produced W or Z pair can subsequently decay into lighter fermions, and the process like $h^0 \to W^*(f\bar{f})W$ contributes considerable amount of Higgs decay, where $W^*(f\bar{f})$ means an off-shell W decaying to a pair of fermions. More details can be found in [8]

(c) Now we come to the process $h^0 \to gg$. The leading order contribution comes from diagrams with one quark loop.



The amplitude reads

$$\begin{aligned}
\mathrm{i}\mathcal{M}(h^0 \to gg) = &-\frac{\mathrm{i}m_q}{v}(\mathrm{i}g_s)^2 \epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\,\mathrm{tr}\,(t^a t^b) \\
&\times \int \frac{\mathrm{d}^d q}{(2\pi)^d}\bigg\{(-1)\,\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not{q} - m_q}\gamma^\nu \frac{\mathrm{i}}{\not{q} + \not{p}_2 - m_q}\frac{\mathrm{i}}{\not{q} - \not{p}_1 - m_q}\right] \\
&+ (-1)\,\mathrm{tr}\left[\gamma^\nu \frac{\mathrm{i}}{\not{q} - m_q}\gamma^\mu \frac{\mathrm{i}}{\not{q} + \not{p}_1 - m_q}\frac{\mathrm{i}}{\not{q} - \not{p}_2 - m_q}\right]\bigg\} \qquad (21.61)
\end{aligned}$$

The first trace in the integrand can be simplified through standard procedure,

$$\begin{aligned}
&\mathrm{tr}\left[\gamma^\mu \frac{\mathrm{i}}{\not{q} - m_q}\gamma^\nu \frac{\mathrm{i}}{\not{q} + \not{p}_2 - m_q}\frac{\mathrm{i}}{\not{q} - \not{p}_1 - m_q}\right] \\
&= \frac{-\mathrm{i}\,\mathrm{tr}\left[(\not{q} + m_q)(\not{q} + \not{p}_2 - m_q)(\not{q} - \not{p}_1 - m_q)\right]}{(q^2 - m_q^2)\left[(q+p_2)^2 - m_q^2\right]\left[(q-p_1)^2 - m_q^2\right]} \\
&= -2\mathrm{i}\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{N^{\mu\nu}}{(q'^2 - \Delta)^3}, \qquad (21.62)
\end{aligned}$$

where

$$q'_\mu = q_\mu - x p_{1\mu} + y p_{2\mu}, \qquad (21.63)$$
$$\Delta = m_q^2 - x(1-x)p_1^2 - y(1-y)p_2^2 - 2xy p_1 \cdot p_2 = m_q^2 - xy m_h^2,$$
$$N^{\mu\nu} = 4m_q\big(p_1^\nu p_2^\nu - p_1^\mu p_2^\nu + 2p_2^\nu q^\mu - 2p_1^\mu q^\nu + 4q^\mu q^\nu + (m_q^2 - p_1\cdot p_2 - q^2)\eta^{\mu\nu}\big) \qquad (21.64)$$

Then we can reexpress $N^{\mu\nu}$ in terms of q', p_1 and p_2 and drop off all terms linear in q which integrates to zero. It is most easy to work with definite helicity states for final gluons. Then the result gets simplified if we dot $N^{\mu\nu}$ with polarization vectors as $N^{\mu\nu}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)$. Note that $\epsilon^*(p_i){\cdot}p_j = 0$ with $i,j=1,2$. Note also the on-shell condition $p_1^2 = p_2^2 = 0,\ p_1{\cdot}p_2 = \frac{1}{2}m_h^2$. then

$$N^{\mu\nu}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) = 4m_q\Big[m_q^2 + \big(xy - \tfrac{1}{2}\big)m_h^2 + \big(\tfrac{4}{d} - 1\big)q'^2\Big]\epsilon^*(p_1)\cdot\epsilon^*(p_2). \tag{21.65}$$

The same calculation shows that the second trace in the integrand of (21.61) gives identical result with the first trace. To check the gauge invariance of this result, one can simply replace $\epsilon^*_\mu(p_1)$ with $p_{1\mu}$ in the expression above, then it is straightforward to find that $N^{\mu\nu}p_{1\mu}\epsilon^*_\nu(p_2) = 0$. Similarly, it can also be checked that $N^{\mu\nu}\epsilon^*_\mu(p_1)p_{2\nu} = 0$.

Then the amplitude (21.61) now reads

$$\mathrm{i}\mathcal{M}(h^0\to gg) = -\frac{2g_s^2 m_q}{v}\delta^{ab}\int_0^1 \mathrm{d}x\int_0^{1-x}\mathrm{d}y\int\frac{\mathrm{d}^d q'}{(2\pi)^d}\frac{N^{\mu\nu}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)}{(q'^2-\Delta)^3}, \tag{21.66}$$

where the relation $\operatorname{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$ in fundamental representation is also used. The momentum integration is finite as $d\to 4$ under dimensional regularization, and can now be carried out directly to be

$$\begin{aligned}\mathrm{i}\mathcal{M}(h^0\to gg) &= -\frac{2\mathrm{i}g_s^2 m_q^2}{(4\pi)^2 v}\delta^{ab}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{(1-4xy)m_h^2}{m_q^2 - xym_h^2}\\ &= -\frac{\mathrm{i}\alpha_s m_h^2}{6\pi v}\delta^{ab}\epsilon^*(p_1)\cdot\epsilon^*(p_2)I_f(\tau_q),\end{aligned} \tag{21.67}$$

where $\tau_q \equiv (m_h/m_q)^2$, and

$$I_f(\tau_q) \equiv 3\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\,\frac{1-4xy}{1-xy\tau_q}$$

Note that the inner product between two polarization vectors is nonzero only for $\epsilon^*_+\cdot\epsilon^*_-$ and $\epsilon^*_-\cdot\epsilon^*_+$. Therefore the squared amplitude with final states polarizations, color indices summed ($\delta^{ab}\delta_{ab} = 8$) is,

$$|\mathcal{M}(h^0\to gg)|^2 = |\mathcal{M}_{+-}(h^0\to gg)|^2 + |\mathcal{M}_{-+}(h^0\to gg)|^2 = \frac{4\alpha_s^2 m_h^4}{9\pi^2 v^2}|I_f(\tau_q)|^2, \tag{21.68}$$

and the decay width is

$$\Gamma(h^0\to gg) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha_s^2}{9\pi^2}\cdot|I_f(\tau_q)|^2, \tag{21.69}$$

where an additional factor 1/2 should be included in (21.51) when calculating $\Gamma(h^0\to gg)$ because the two gluons in final states are identical particles. This result is easily generalized for N_q copies of quarks to be

$$\Gamma(h^0\to gg) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_W^2}\cdot\frac{\alpha_s^2}{9\pi^2}\cdot\Big|\sum_q I_f(\tau_q)\Big|^2, \tag{21.70}$$

(d) Now we calculate the cross section for the Higgs production via gluon fusion at the leading order. The amplitude is simply given by the result in (c), namely (21.67). When we take the square of this amplitude, an additional factor $(\frac{1}{8}\cdot\frac{1}{2})^2$ should be included, to average over helicities and color indices of initial gluons. Then, comparing (21.52) with (21.51), we find that

$$\sigma(gg\to h^0)=\frac{\pi^2}{8m_h}\delta(\hat{s}-m_h^2)\Gamma(h^0\to gg), \tag{21.71}$$

where the hatted variable $\hat{s}$ is the parton level center-of-mass energy. We note again that the correct formula is obtained by including an factor of $(\frac{1}{8}\cdot\frac{1}{2})^2$ in $\sigma(gg\to h^0)$ to average over the initial degrees of freedom of two gluons, and an factor of 1/2 in $\Gamma(h^0\to gg)$ to count the identical particles in the final state. Then, from (21.70), it is straightforward to find

$$\sigma(gg\to h^0)=\frac{\alpha\alpha_s^2}{576\sin^2\theta_w}\cdot\frac{m_h^2}{m_W^2}\Big|\sum_q I_f(\tau_q)\Big|^2\delta(\hat{s}-m_h^2). \tag{21.72}$$

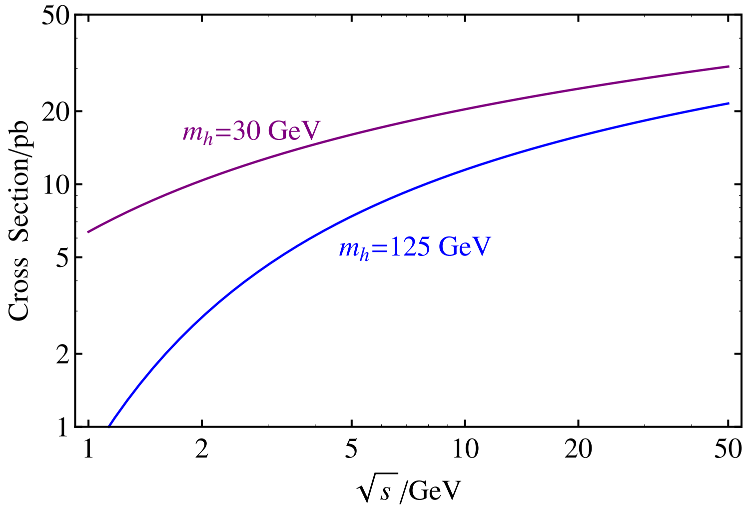
Then the proton-level cross section of Higgs boson production via gluon-gluon fusion is given by

$$\begin{aligned}&\sigma_{GGF}\big(p(P_1)p(P_2)\to h^0\big)\\ &=\int_0^1 \mathrm{d}x_1\int_0^1\mathrm{d}x_2\, f_g(x_1)f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big)\\ &=\int \mathrm{d}M^2\mathrm{d}Y\left|\frac{\partial(x_1,x_2)}{\partial(M^2,Y)}\right|f_g(x_1)f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big)\\ &=\int \mathrm{d}M^2\mathrm{d}Y\,\frac{1}{M^2}x_1f_g(x_1)x_2f_g(x_2)\sigma\big(g(x_1P_1)g(x_2P_2)\to h^0\big),\end{aligned} \tag{21.73}$$

where $M^2=x_1x_2s$ is the center-of-mass energy of two initial gluons, while s is the center-of-mass energy of two initial protons, and Y, given by $\exp Y=\sqrt{x_1/x_2}$, is the rapidity of the produced Higgs boson relative to the center-of-mass frame of the proton system. (Note that in our case $M^2=m_h^2$.) The relations between M^2, Y and the momentum fractions x_1, x_2 can be inverted to give $x_1=(M/\sqrt{s})e^Y$ and $x_2=(M/\sqrt{s})e^{-Y}$. Furthermore, f_g is the parton distribution function of the gluon in a proton, which we will take to be $f_g=8(1-x)^7/x$ in the following calculations. Then the cross section can be evaluated to be

$$\begin{aligned}&\sigma_{GGF}\big(p(P_1)p(P_2)\to h^0\big)\\ &=\frac{\alpha\alpha_s^2}{9\sin^2\theta_w}\cdot\frac{1}{m_W^2}\Big|\sum_q I_f(\tau_q)\Big|^2\int_{-Y_0}^{Y_0}\mathrm{d}Y\left(1-\frac{m_h}{\sqrt{s}}e^Y\right)^7\left(1-\frac{m_h}{\sqrt{s}}e^{-Y}\right)^7,\end{aligned} \tag{21.74}$$

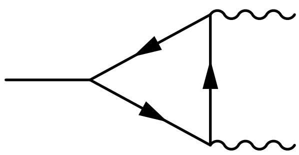
where Y_0, given by $\cosh Y_0=\sqrt{s}/2m_h$ is the largest possible rapidity of a produced Higgs boson. We plot this cross section as a function of the center-of-mass energy $\sqrt{s}$ of the pp pair, with the Higgs boson's mass taken to be $m_h=30$GeV and $m_h=125$GeV, respectively, in Figure



(e) Next we consider the process $h^0 \to 2\gamma$. The contribution to this decay channel at the leading (1-loop) level is from two types of diagrams, one with a fermion loop and the other with a W boson (and related would-be Goldstone boson) loop. The former contribution is easy to find by virtue of the result in (c) for $h^0 \to gg$. The calculation here is in fully parallel, except that we should include the factor for the electric charges of internal fermions Q_f, take away the color factor $\operatorname{tr}(t^a t^b)$, change the strong coupling g_s by the electromagnetic coupling e, and sum over all charged fermions. Note that the color factor enters the expression of the decay width as $|\operatorname{tr}(t^a t^b)|^2 = \frac{1}{2}\delta^{ab}\frac{1}{2}\delta^{ab} = 2$, then it is straightforward to write down the fermion contribution to the $h^0 \to 2\gamma$ to be

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_f = \left(\frac{\alpha m_h}{8\sin^2\theta_w}\right)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha_s^2}{18\pi^2}\cdot\Big|\sum_f Q_f^2 N_c(f) I_f(\tau_f)\Big|^2, \tag{21.75}$$

where $N_c(f)$ is the color factor, equal to 3 for quarks and 1 for charged leptons.



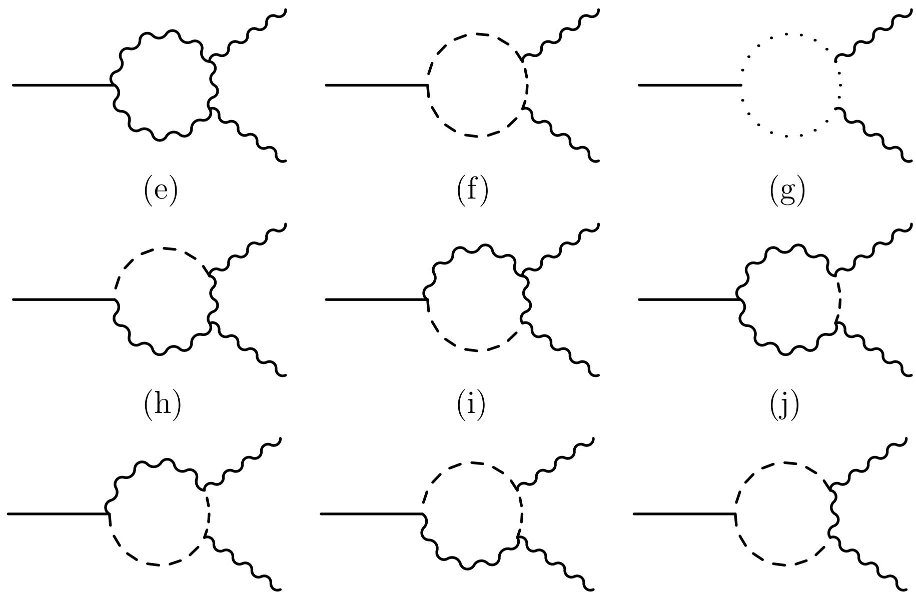
(f) Now we come to the W-loop contributions to $h^0 \to 2\gamma$. In Feynman-'t Hooft gauge, we should also include the corresponding Goldstone loop diagrams. Then there are 13 diagrams in total. We compute them as follows,

(a) (b)

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{a})} &= \frac{1}{2}\frac{\mathrm{i}g_{\rho\sigma}g^2 v}{2}(-\mathrm{i}e^2)(2\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2) \\
&\quad \times \int \frac{\mathrm{d}^d q}{(2\pi)^d} D_W(q) D_W(k-q) \\
&= -\frac{2\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2 m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)(d-1)\Gamma(2-\tfrac{d}{2}) \\
&\quad \times \int_0^1 \frac{\mathrm{d}x}{[m_W^2 - x(1-x)m_h^2]^{2-d/2}}, \qquad (21.76) \\
\mathrm{i}\mathcal{M}^{(\mathrm{b})} &= \frac{1}{2}(-2\mathrm{i}\lambda v)(2\mathrm{i}e^2)\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int \frac{\mathrm{d}^d q}{(2\pi)^d} D_s(q) D_s(k-q) \\
&= -\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2 m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2}) \\
&\quad \times \int_0^1 \frac{\mathrm{d}x}{[m_W^2 - x(1-x)m_h^2]^{2-d/2}}. \qquad (21.77)
\end{aligned}
$$

(c) (d)

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{c})} = \mathrm{i}\mathcal{M}^{(\mathrm{d})} &= \frac{\mathrm{i}g^2\sin\theta_w}{2}\cdot\frac{\mathrm{i}g^2 v\sin\theta_w}{2}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_s(q)D_W(p_2-q) \\
&= -\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2 m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\frac{1}{(m_W^2)^{2-d/2}}. \qquad (21.78)
\end{aligned}
$$



(e) (f) (g)

(h) (i) (j)

(k) (l) (m)

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{e})} = & \frac{\mathrm{i}g^2 v}{2}(-\mathrm{i}e)^2 \eta_{\rho\sigma}\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_W(q)D_W(q-p_1)D_W(q+p_2)\\
&\times\left[\eta^{\rho\lambda}(2q-p_1)^\mu+\eta^{\mu\rho}(2p_1-q)^\lambda-\eta^{\lambda\mu}(p_1+q)^\rho\right]\\
&\times\left[\eta^\sigma_\lambda(2q+p_2)^\nu-\eta^\nu_\lambda(q-p_2)^\sigma-\eta^{\sigma\nu}(2p_2+q)_\lambda\right]\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\bigg[\int\mathrm{d}x\mathrm{d}y\frac{(5-x-y+4xy)m_h^2}{m_W^2-xym_h^2}\\
&+6(d-1)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}}\bigg],
\end{aligned} \tag{21.79}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{f})} = & (-2\mathrm{i}\lambda\nu)(-\mathrm{i}e)^2\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(2q-p_1)^\mu(2q+p_2)^\nu\\
&\times D_s(q)D_s(q-p_1)D_s(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{2\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned} \tag{21.80}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{g})} = & \Big(-\frac{\mathrm{i}m_W^2}{v}\Big)(\mathrm{i}e)^2\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(-1)(q-p_1)^\mu q^\nu\\
&\times D_s(q)D_s(q-p_1)D_s(q+p_2)\\
=&-\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned} \tag{21.81}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{h})} = \mathrm{i}\mathcal{M}^{(\mathrm{i})} = & \frac{\mathrm{i}g}{2}\frac{\mathrm{i}g^{\mu\lambda}g^2v\sin\theta_w}{2}(-\mathrm{i}e)\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\int\frac{\mathrm{d}^d q}{(2\pi)^d}(q-p_1-k)^\sigma\\
&\times\left[\eta_{\sigma\lambda}(2q+p_2)^\nu-\eta^\nu_\lambda(q-p_2)_\sigma-\eta^\nu_\sigma(2p_2+q)_\lambda\right]\\
&\times D_W(q)D_s(q-p_1)D_W(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\bigg[\int\mathrm{d}x\mathrm{d}y\frac{(1-x)(1+y)m_h^2}{m_W^2-xym_h^2}\\
&-\frac{1}{2}(d-1)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}}\bigg],
\end{aligned} \tag{21.82}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{j})} = & \frac{\mathrm{i}g^2 v}{2}\Big(\frac{\mathrm{i}g^2v\sin\theta_w}{2}\Big)^2\epsilon^*(p_1)\cdot\epsilon^*(p_2)\\
&\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_s(q)D_W(q-p_1)D_W(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\mathrm{d}x\mathrm{d}y\frac{2m_W^2}{m_W^2-xym_h^2},
\end{aligned} \tag{21.83}$$

$$\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{k})} = \mathrm{i}\mathcal{M}^{(\mathrm{l})} = & \frac{\mathrm{i}g}{2}\frac{\mathrm{i}g^2v\sin\theta_w}{2}(-\mathrm{i}e)\epsilon^*_\mu(p_1)\epsilon^*_\nu(p_2)\\
&\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}(p_1+2p_2+q)^\mu(2q+p_2)^\nu D_s(q)D_W(q-p_1)D_s(q+p_2)\\
=&\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Gamma(2-\tfrac{d}{2})\int\frac{\mathrm{d}x\mathrm{d}y}{(m_W^2-xym_h^2)^{2-d/2}},
\end{aligned} \tag{21.84}$$

$$
\begin{aligned}
\mathrm{i}\mathcal{M}^{(\mathrm{m})} &= (-2\mathrm{i}\lambda\nu)\Big(\frac{\mathrm{i}g^2 v\sin\theta_w}{2}\Big)^2\epsilon^*(p_1)\cdot\epsilon^*(p_2) \\
&\quad\times\int\frac{\mathrm{d}^d q}{(2\pi)^d}D_W(q)D_s(q-p_1)D_s(q+p_2) \\
&= \frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_h^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\int\mathrm{d}x\mathrm{d}y\frac{m_W^2}{m_W^2-xym_h^2}. \qquad (21.85)
\end{aligned}
$$

The results can be summarized as,

$$
\mathrm{i}\mathcal{M}^{(X)} = \frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{e^2m_W^2}{v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Big[A\cdot\Gamma(2-\tfrac{d}{2})+B\Big], \quad (X=a,b,\cdots,m) \qquad (21.86)
$$

with the coefficients A and B for each diagram listed in Table.

Diagrams	A	B
(a)	$-2(d-1)J_1$	0
(c)+(d)	$-2(m_W^2)^{d/2-2}$	0
(e)	$6(d-1)J_2$	J_3
(g)	$-J_2$	0
(h)+(i)	$-(d-1)J_2$	$2J_4$
(j)	0	$2(m_W/m_h)^2J_5$
(k)+(l)	$2J_2$	0
(m)	0	J_5
(b)	$-(m_h/m_W)^2J_1$	0
(f)	$2(m_h/m_W)^2J_2$	0

where

$$
\begin{aligned}
J_1 &= \int_0^1\mathrm{d}x\frac{1}{[m_W^2-x(1-x)m_h^2]^{2-d/2}} \\
&= 1-\frac{\epsilon}{2}\int_0^1\mathrm{d}x\log\Big(m_W^2-x(1-x)m_h^2\Big)+\mathcal{O}(\epsilon^2), \qquad (21.87)\\
J_2 &= \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{1}{(m_W^2-xym_h^2)^{2-d/2}} \\
&= \frac{1}{2}-\frac{\epsilon}{2}\int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\log\Big(m_W^2-xym_h^2\Big)+\mathcal{O}(\epsilon^2), \qquad (21.88)\\
J_3 &= \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(5-x-y+4xy)m_h^2}{m_W^2-xym_h^2}, \qquad (21.89)\\
J_4 &= \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{(1-x)(1+y)m_h^2}{m_W^2-xym_h^2}, \qquad (21.90)\\
J_5 &= \int_0^1\mathrm{d}x\int_0^{1-x}\mathrm{d}y\frac{m_h^2}{m_W^2-xym_h^2}. \qquad (21.91)
\end{aligned}
$$

To see that the divergences of all diagrams cancel among themselves, it just needs to show that sum of all A-coefficients is of order ϵ. This is straightforward by noting that $J_1 = 1+\mathcal{O}(\epsilon)$ and $J_2 = 1/2+\mathcal{O}(\epsilon)$.

Before reaching the complete result, let us first find out the W-loop contribution in the limit $m_h^2 \ll m_W^2$, although it seems unlikely to be true within our current knowledge. To find the amplitude in this limit, we expand the five integrals $J_1, \cdots, J_5$ in terms of m_h/m_W,

$$J_1 \simeq 1 - \frac{\epsilon}{2}\log m_W^2 + \frac{\epsilon}{12}\frac{m_h^2}{m_W^2}, \qquad J_2 \simeq \frac{1}{2} - \frac{\epsilon}{4}\log m_W^2 + \frac{\epsilon}{48}\frac{m_h^2}{m_W^2},$$

$$J_3 \simeq \frac{7}{3}\frac{m_h^2}{m_W^2}, \qquad J_4 \simeq \frac{11}{24}\frac{m_h^2}{m_W^2}, \qquad J_5 \simeq \frac{1}{2}\frac{m_h^2}{m_W^2} + \frac{1}{24}\Big(\frac{m_h^2}{m_W^2}\Big)^2.$$

Then the amplitude can be recast into

$$\mathrm{i}\mathcal{M} = \frac{\mathrm{i}e^2 m_W^2}{(4\pi)^2 v}\epsilon^*(p_1)\cdot\epsilon^*(p_2)\Big[C\Big(\frac{2}{\epsilon} - \gamma + \log 4\pi\Big) + D\cdot\log m_W^2 + E + F\cdot\frac{m_h^2}{m_W^2}\Big] \tag{21.92}$$

Diagrams	C	D	E	F
(a)	-6	3	4	-1
(c)+(d)	-2	1	0	0
(e)	9	$-9/2$	-6	$37/12$
(g)	$-1/2$	$1/4$	0	$-1/24$
(h)+(i)	$-3/2$	$3/4$	1	$19/24$
(j)	0	0	1	$1/12$
(k)+(l)	1	$-1/2$	0	$1/12$
(m)	0	0	0	$1/2$
(b)	$-(m_h/m_W)^2$	$(m_h/m_W)^2/2$	0	0
(f)	$(m_h/m_W)^2$	$-(m_h/m_W)^2/2$	0	0
sum	0	0	0	$7/2$

Therefore, the amplitude in the limit $m_h^2 \ll m_W^2$ is given by

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_W = 2\cdot\frac{7}{2}\frac{\mathrm{i}\alpha m_h^2}{4\pi v}\epsilon^*(p_1)\cdot\epsilon^*(p_2), \tag{21.93}$$

where the factor 2 counts the identical contributions from the diagrams with two final photons changed. Now we sum up the fermion-loop contribution found in (e) and the result here to get the $h^0 \to 2\gamma$ amplitude in the light Higgs limit,

$$\mathrm{i}\mathcal{M} = -\frac{\mathrm{i}\alpha m_h^2}{3\pi v}\Big[\sum_f Q_f^2 N_c(f) - \frac{21}{4}\Big]\epsilon^*(p_1)\cdot\epsilon^*(p_2). \tag{21.94}$$

Then the corresponding partial width is given by

$$\Gamma(h^0 \to 2\gamma) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big)\cdot\frac{m_h^2}{m_w^2}\cdot\frac{\alpha^2}{18\pi^2}\cdot\Big|\sum_f Q_f^2 N_c(f) - \frac{21}{4}\Big|^2, \tag{21.95}$$

Now we retain m_h as a free variable. Then the various diagrams sum into the following full expression for the W-loop contribution to $h^0 \to 2\gamma$,

$$\mathrm{i}\mathcal{M}(h^0 \to 2\gamma)_W = \frac{\mathrm{i}\alpha m_h^2}{2\pi v}\epsilon^*(p_1)\cdot\epsilon^*(p_2) I_W(\tau_W), \tag{21.96}$$

where the factor $I_W(\tau_W)$, as a function of $\tau_W \equiv (m_h/m_W)^2$, is given by

$$I_W(\tau_W) = \frac{1}{\tau_W}\Big[6I_1(\tau_W) - 8I_2(\tau_W) + \tau_W\big(I_1(\tau_W) - I_2(\tau_W)\big) + I_3(\tau_W)\Big], \tag{21.97}$$

where

$$I_1(\tau_W) \equiv \int_0^1 \mathrm{d}x\, \log\big[1 - x(1-x)\tau_W\big], \tag{21.98}$$

$$I_2(\tau_W) \equiv 2\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \log\big(1 - xy\tau_W\big), \tag{21.99}$$

$$I_3(\tau_W) \equiv \int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{d}y\, \frac{(8 - 3x + y + 4xy)\tau_W}{1 - xy\tau_W}. \tag{21.100}$$

Then the full expression for the partial width of $h^0 \to 2\gamma$ at one-loop is

$$\Gamma(h^0 \to 2\gamma) = \Big(\frac{\alpha m_h}{8\sin^2\theta_w}\Big) \cdot \frac{m_h^2}{m_w^2} \cdot \frac{\alpha^2}{18\pi^2} \cdot \Big|\sum_f Q_f^2 N_c(f) I_f(\tau_f) - I_W(\tau_W)\Big|^2, \tag{21.101}$$

(h) Collecting all results above (expect the $\gamma\gamma$ channel, which is quite small*), we plot the total width and decay branching fractions of the Higgs boson in Figures 21.6 and 21.7, respectively.

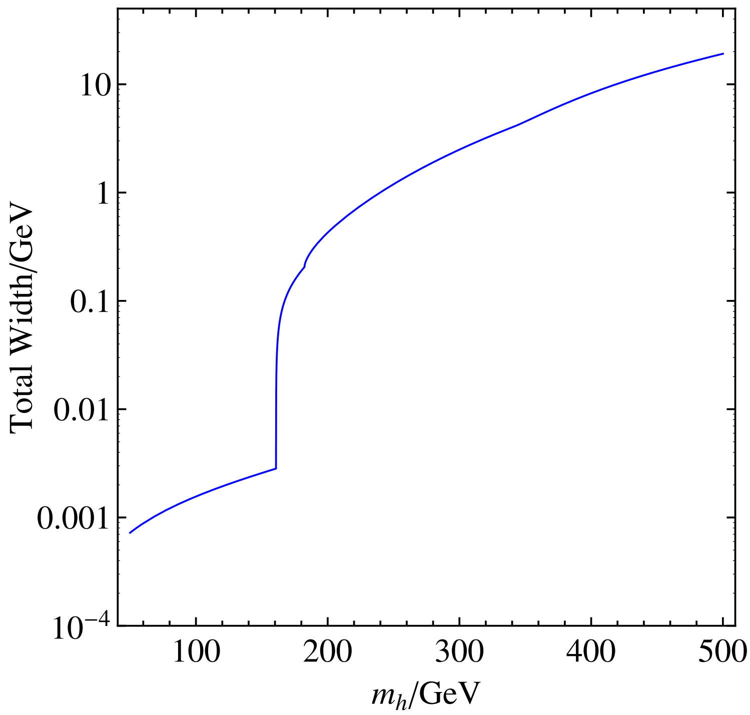


Figure 21.6: The total width of the Higgs boson as a function of its mass.

*– but very important!

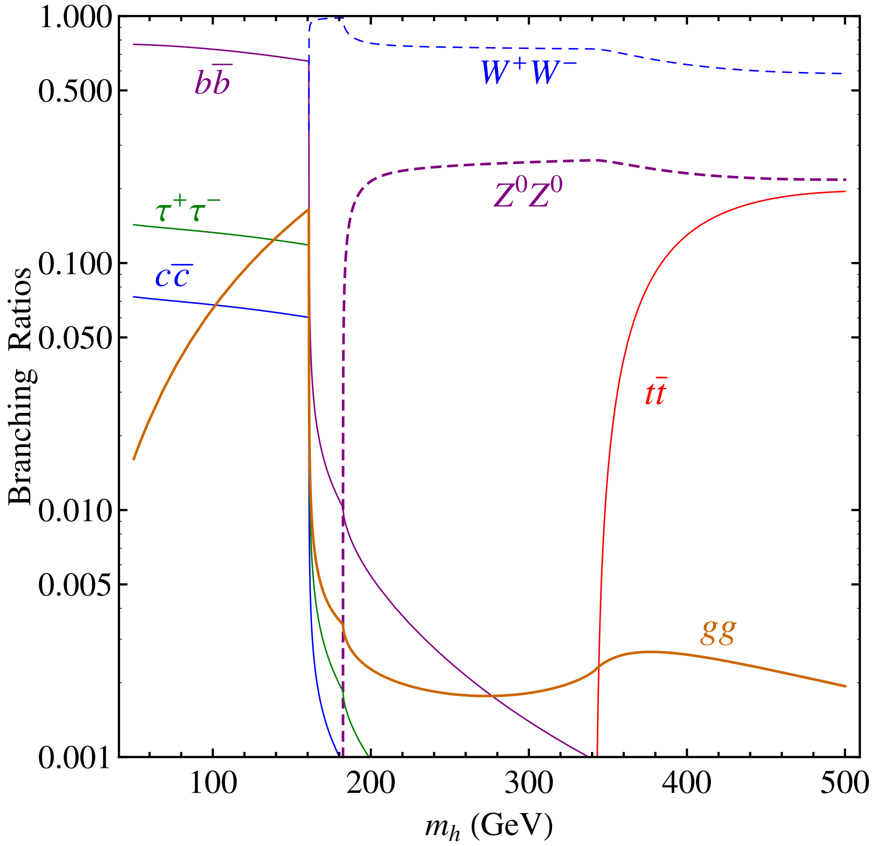


Figure 21.7: The Higgs decay branching fractions of $t\bar{t}$, $b\bar{b}$, $c\bar{c}$, $\tau^+\tau^-$, WW, ZZ and gg channels, as functions of Higgs mass.

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